Molecular Motors, Brownian Ratchets, and Reflected Diffusions.

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Abstract

Molecular motors are protein structures that play a central role in accomplishing mechanical work inside a cell. While chemical reactions fuel this work, it is not exactly known how this chemical-mechanical conversion occurs. Recent advances in microbiological techniques have enabled at least indirect observations of molecular motors which in turn have led to significant effort in the mathematical modeling of these motors in the hope of shedding light on the underlying mechanisms involved in intracellular transport. Kinesin which moves along microtubules that are spread throughout the cell is a prime example of the type of motors that are studied in this work. The motion is linked to the presence of a chemical, ATP, but how the ATP is involved in motion is not clearly understood. One commonly used model for the dynamics of Kinesin in the biophysics literature is the Brownian ratchet mechanism. In this work, we give a precise mathematical formulation of a Brownian ratchet (or more generally a diffusion ratchet) via an infinite system of stochastic differential equations with reflection. This formulation is seen to arise in the weak limit of a natural discrete space model that is often used to describe motor dynamics in the literature. Expressions for asymptotic velocity and effective diffusivity of a biological motor modeled via a Brownian ratchet are obtained. Linearly progressive biomolecular motors

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often carry cargos via an elastic linkage. A two-dimensional coupled stochastic dynamical system is introduced to model the dynamics of the motor-cargo pair. By proving that an associated two dimensional Markov process has a unique stationary distribution, it is shown that the asymptotic velocity of a motor pulling a cargo is well defined as a certain Law of Large Number limit, and finally an expression for the asymptotic velocity in terms of the invariant measure of the Markov process is obtained.

1 Introduction.

Molecular motors are proteins, or structures consisting of multiple proteins, that play a central role in accomplishing mechanical work in the interior of a living cell. Frequently, the exact nature of the chemical-mechanical energy conversion is not completely understood. However, recent advances in molecular biology have enabled *in vitro* observations of molecular motors and/or their cargos which in turn have led to significant research in the mathematical modeling of these motors in the hope of shedding light on the underlying mechanisms involved in intracellular transport.

The motors which we have in mind for this paper, such as kinesin, have multiple heads which move along a microtubule–stepping in a hand over hand manner. One head remains fixed while the other diffuses into the next binding site. Then, the previously fixed head is released and can diffuse to the next binding site, etc. Extending from this dual head structure is a long protein strand which can attach to a cargo.

One commonly studied model for the dynamics of a molecular motor is the *Brownian Ratchet* model. In a Brownian ratchet, a particle representing the biological motor diffuses between equally spaced barriers. When the particle encounters the barrier from the left it is free to pass through; however, it is instantaneously reflected back when it encounters the barrier from the right. Hence, the ratchet mechanism has the effect of introducing a positive drift to the dynamics of the particle. This ratcheting mechanism models the binding of the motor head to the microtubule. In practice, one is interested in gaining information about the asymptotic velocity of the motor, first passage times, locations and distances between barrier sites, parameters of the governing diffusion, etc.

Our first goal in this study is to give a precise mathematical formulation of the stochastic process that gives a pathwise representation for such a ratcheting mechanism. We will begin by considering a pure Jump-Markov process that captures the dynamical description given above. In view of the fact that the temporal and spatial scales at which the motor is observed are typically much larger than the step sizes and mean time intervals between successive steps, we consider diffusion approximation of the above pure Jump-Markov process by suitable scaling. By using weak convergence methods we obtain a $I\!R^+$ valued stochastic process with continuous sample paths, which we call the "Diffusion Ratchet", that arises as the diffusion limit of the above Jump-Markov process. This stochastic process, denoted as $\{X(t)\}$ will be described via an infinite system of stochastic differential equations with reflection. Next, we will consider two quantities associated with the model which are of great practical interest: Asymptotic velocity and effective diffusivity. Using some basic renewal theory we will obtain an a.s. deterministic limit, ν, for $\frac{X(t)}{t}$ as $t \to \infty$. This Law of Large Number(LLN) limit is referred to as the *asymptotic velocity* of the motor. We will also obtain a Functional Central Limit theorem for $\xi_n(t) \doteq \frac{X(nt)-\nu nt}{\sqrt{nt}}$ showing that ξ_n converges weakly in $C([0,\infty) : I\!R)$ to σW, where W is a standard Wiener process. The constant σ^2 is referred to as the *effective diffusivity* of the motor.

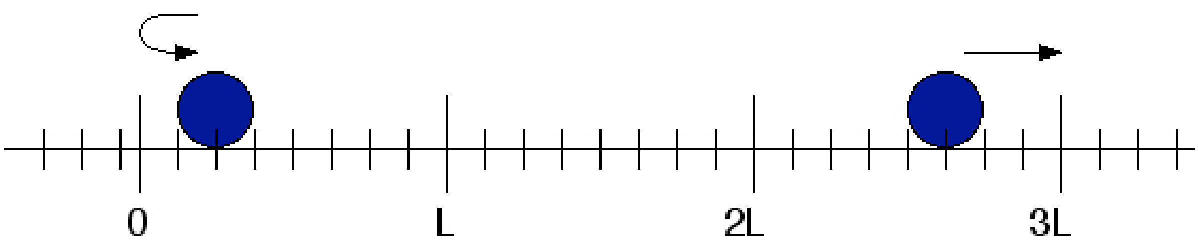
Biological motors are typically responsible for intracellular transport of cargoes, such as large protein molecules, to locations in the cell where they are needed. Unlike the ratchet process which models the dynamics of the motor, the process representing the cargo has no reflecting barriers since the cargo is floating somewhat freely in the cell, attached to the motor via a protein strand. However, there is interaction between the two processes. The farther the cargo is behind the motor the greater the forward drift for the cargo and the greater the backward drift of the motor. Since the motor is moving along a straight track ($I\!R^+$), which without loss of generality can be taken to be the x-axis, only the dynamics of the x-coordinate of the location of the cargo is coupled with the motor dynamics. Denote the location of the motor and the x-coordinate of the cargo at time instant t as $X(t)$ and $Y(t)$, respectively. In Section 3, we will model the pair, $(X(\cdot), Y(\cdot))$, as a stochastic process with paths in $C([0,\infty) : I\!R^+ \times I\!R)$ given via an infinite system of partially reflected two dimensional diffusion processes. In order to justify the model, we will once again consider the natural Jump-Markov process that captures the dynamical description given above and prove that, after appropriate scaling, the Jump-Markov process converges weakly (in a suitable function space) to (X, Y).

Finally in Section 4, we will undertake the study of asymptotic velocity for the case where the motor is pulling a cargo. The renewal theory arguments, that rely on the underlying independence of the motor dynamics between different barriers, break down in the coupled motor-cargo case. Due to this difficulty, we will consider the special case where the interaction between cargo and motor is modeled via a linear spring. Denoting the length of the elastic linkage, $X(t) - Y(t)$, by $Z(t)$. We will show in Proposition 4.4 that the Markov process $\Pi(t) \doteq (\lfloor X(t)/L \rfloor L, Z(t))$, where L denotes the distance between successive barriers, has a unique invariant probability measure. This result will show that $\frac{1}{t}\int_0^t Z(s)ds$ converges in probability to a deterministic quantity which is independent of the initial conditions, and $\frac{Z(t)}{t}$ converges in probability to zero. As an immediate consequence, we get that $\frac{X(t)}{t}$ converges in probability to a deterministic quantity (Theorem 4.5). This limiting quantity is defined to be the asymptotic velocity of the motor and can be expressed in terms of the invariant distribution of $\Pi(t)$.

The paper is organized as follows. In Section 2 we will consider the model for the "Motor only" case. We will present here the key diffusion approximation result and also study the asymptotic velocity and effective diffusivity of the motor. Section 3 considers the coupled motor-cargo system. The main result of this section is the weak convergence theorem that proves the convergence of the natural Jump-Markov process that is associated with the dynamics of the motor-cargo pair to a pair of stochastic processes with continuous sample paths. Finally in Section 4 we will study the asymptotic velocity of the motor-cargo pair. Some notation that will be used in this paper is as follows. $I\!R$ and $I\!R^+$ will denote the space of reals and non-negative reals, respectively. The space of positive (non-negative) integers will be denoted by $I\!N$ (resp. $I\!N_0$). The space of integers will be denoted by $\mathbb{Z}$. For a metric space E, the space of continuous (resp. right continuous with left limits) functions from $[0, \infty)$ to E will be denoted by $C([0, \infty) : E)$ (resp. $D([0, \infty) : E)$). Borel σ-field for a Polish space E will be denoted by $\mathcal{B}(E)$.

2 Mathematical Formulation of a Diffusion Ratchet.

In this section we begin by describing the dynamics of a biological motor which is moving along a linear track following a "ratchet mechanism". We then introduce a discrete state Jump-Markov process that captures the described dynamics. By suitable scaling, we consider a diffusion approximation for this Jump-Markov process and in the limit obtain a $I\!R^+$ valued stochastic process with continuous sample paths that we call the "Diffusion Ratchet". Finally, using some basic renewal theory, we obtain the asymptotic velocity and effective diffusivity for the motor that is predicted by the diffusion ratchet model.



Consider a particle, representing a biological motor moving on a track positioned along the x-axis. Ratchet sites, $\mathbb{L} \doteq \{iL : i \in I\!N_0\}$, are located on the track at equally spaced intervals of length L. When the particle is at a "non-ratchet" site it can move either to the left or to the right. However, when the particle is at a ratchet site, it can move only to the right. As stated earlier, the step sizes of the motor are much smaller in comparison to the spatial scales at which observations are made. We will assume that the motor takes steps of size $\frac{1}{n}$, where n represents a scaling parameter and is assumed to belong to $I\!N' \doteq \{\frac{m}{L}, m \in I\!N\}$. The choice of $I\!N'$ in defining the step sizes rather than $I\!N$ is made for convenience. It assures that the ratchet sites are on the discrete lattice $\mathbb{S}_n \doteq \{\frac{j}{n}, j \in I\!N_0\}$ for all $n \in I\!N'$. We model the above described dynamics by a Jump-Markov process (cf. [6], Chapter 7) with RCLL (right continuous with left limits) paths, $\{X_n(t)\}_{t \geq 0}$ where $X_n(t)$ represents the location of the particle at time t. The infinitesimal generator of the Markov process, denoted by $\mathfrak{q}_n \equiv \{\mathfrak{q}_n(x, y) : x, y \in \mathbb{S}_n\}$, is given as follows. For $x, y \in \mathbb{S}_n$, $x \neq y$:

$$\mathfrak{q}_n(x,y) \doteq \begin{cases} n^2\alpha(x) + n\beta_1(x), & y = x + \frac{1}{n}, \\ (n^2\alpha(x) + n\beta_2(x))1_{x\in\mathbb{S}_n\setminus\mathbb{L}} & y = x - \frac{1}{n}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

Also, for $x \in \mathbb{S}_n$, $\mathfrak{q}_n(x,x) \doteq -(\mathfrak{q}_n(x, x+\frac{1}{n}) + \mathfrak{q}_n(x, x-\frac{1}{n}))$. Here, $\alpha, \beta_1, \beta_2 : I\!R^+ \to I\!R$ are assumed to be globally Lipschitz. The generator can be interpreted as follows. Given that the motor is at x at some time instant, it spends a sojourn time which is exponentially distributed with rate $\lambda_n(x) \doteq -\mathfrak{q}_n(x,x)$. If x is a non-ratchet site, at the end of sojourn time, it moves to either $x+\frac{1}{n}$ (with probability $p_n(x) = \frac{n\alpha(x)+\beta_1(x)}{2n\alpha(x)+\beta_1(x)+\beta_2(x)}$), or it moves to $x-\frac{1}{n}$ (with probability $1-p_n(x)$). If x is a ratchet site, the particle moves to $x+\frac{1}{n}$, with probability one, at the end of the sojourn time.

We next introduce the diffusion ratchet $X(t)$, which is a stochastic process with continuous sample paths given as follows. Roughly speaking, $X(t)$ behaves like a reflecting diffusion when it is in the interval $[iL, (i+1)L)$; $i \in I\!N_0$, with iL acting as the reflecting barrier. Let $\{W^{(i)}\}_{i\in I\!N_0}$ be a sequence of independent standard Brownian motions given on some probability space $(\Omega, \mathcal{F}, I\!P)$. Denote by $\mathcal{D}_i$ the subset of $\mathcal{D}([0,\infty) : I\!R)$ defined as

$$\mathcal{D}_i \doteq \{x \in \mathcal{D}([0,\infty) : I\!R) | x(0) = iL\}.$$

Also, let

$$\hat{\mathcal{D}}_i \doteq \{x \in \mathcal{D}\big([0,\infty) : [iL,\infty)\big) | x(0) = iL\}.$$

Let $\Gamma_i : \mathcal{D}_i \to \hat{\mathcal{D}}_i$ be the Skorokhod map defined as:

$$\Gamma_i(x)(t) \doteq x(t) - \left(\inf_{0\le s\le t} \big(x(s) - iL\big) \wedge 0\right). \tag{2.2}$$

Let $X^{(i)}(\cdot)$ be the unique strong solution of the integral equation (cf Section 1.2 of [1]):

$$X^{(i)}(t) = \Gamma_i\left(iL + \int_0^{\cdot} b(X^{(i)}(s))ds + \int_0^{\cdot} a(X^{(i)}(s))dW^{(i)}(s)\right)(t),\ t \in [0,\infty), \tag{2.3}$$

where in addition to the Lipschitz condition on the coefficients, we assume that there exist b^*, a_* and a^* in $I\!R$ such that for all $x \in I\!R^+$

$$|b(x)| \le b^* \ \text{ and } \ 0 < a_* \le a(x) \le a^*. \tag{2.4}$$

Next, for $i \in I\!N_0$, define stopping times $\tau^{(i)}$ as

$$\tau^{(i)} \doteq \inf\{t : X^{(i)}(t) \ge (i+1)L\}. \tag{2.5}$$

Also set $\sigma^{(0)} = 0$ and define

$$\sigma^{(i)} \doteq \tau^{(i-1)} + \sigma^{(i-1)}, \ \ i \ge 1. \tag{2.6}$$

The following lemma will guarantee that the diffusion ratchet that we construct has paths in the space $C([0,\infty) : I\!R_+)$.

Lemma 2.1. *For all $i \in I\!N$, $\sigma^{(i)} \in (0, \infty)$ almost surely, and $\sigma^{(i)} \to \infty$ almost surely as $i \to \infty$.*

Proof. In order to show $I\!P(0 < \sigma^{(i)}) = 1$ it suffices to show that $I\!P(0 < \tau^{(0)}) = 1$. Note that $I\!P(X^{(0)}(0) = 0) = 1$, and so $I\!P(X^{(0)}(0) = L) = 0$. The continuity of sample paths of $X^{(0)}(\cdot)$ then implies that and $I\!P(0 < \tau^{(0)}) = 1$.

In order to show $I\!P(\sigma^{(i)} < \infty) = 1$, it suffices to show that $I\!P(\tau^{(j)} < \infty) = 1$ for all j. However, this is an immediate consequence of Theorem 5.1 which in fact says that $I\!E\tau^{(j)} < \infty$ (see appendix). This shows that $I\!P(0 < \sigma^{(i)} < \infty) = 1$ for all i.

For the second part of the lemma, in view of Borel-Cantelli Lemma, it suffices to show that there exists $\delta, \epsilon \in (0, 1)$ such that

$$\inf_{j \in I\!N_0} I\!P(\tau^{(j)} > \delta) \; > \; \epsilon. \tag{2.7}$$

Let $\epsilon \in (0, 1)$ be arbitrary. Define

$$U^{(i)}(u) = iL + \int_0^u b(X^{(i)}(s))ds + \int_0^u a(X^{(i)}(s))dW^{(i)}(s), \;\; u \in [0, \infty). \tag{2.8}$$

Note that for $\delta \in (0, 1)$

$$\begin{aligned} I\!P(\tau^{(j)} \le \delta) &= I\!P(\sup_{0 \le s \le \delta} |X^{(j)}(s) - jL| \ge L) \\ &\le I\!P(\sup_{0 \le s \le \delta} |U^{(j)}(s) - jL| \ge \frac{L}{2}) \\ &\le 2\frac{I\!E(\sup_{0 \le s \le \delta} |U^{(j)}(s) - jL|)}{L} \\ &\le \frac{C\delta^{1/2}}{L}, \end{aligned} \tag{2.9}$$

for a universal constant C, where the last step follows on using (2.4). Now (2.7) follows on choosing δ small enough. ■

We are now ready to define the Diffusion ratchet.

Definition 2.2 (Diffusion Ratchet). *Let, for $i \in I\!N_0$, $X^{(i)}, \tau^{(i)}, \sigma^{(i)}$ be defined via (2.3), (2.5) and (2.6), respectively. A diffusion ratchet is the stochastic process $X(\cdot)$ with paths in $C([0, \infty) : I\!R_+)$ defined as follows.*

$$X(t) \doteq X^{(i)}(t - \sigma^{(i)}); \;\; t \in [\sigma^{(i)}, \sigma^{(i+1)}), \;\; i \in I\!N_0.$$

Note that the process X has the desired properties; namely, after the process has reached iL and before it hits $(i+1)L$, it behaves as a reflected diffusion with drift coefficient b, diffusion coefficient a and the reflecting barrier at iL, .

The following result gives the weak convergence of the Jump-Markov process with generator $\mathfrak{q}_n$ to the diffusion ratchet introduced above. Let α, β_1, β_2 be as in (2.1). Set $b(x) = \beta_1(x) - \beta_2(x)$ and $\alpha(x) = \frac{a^2(x)}{2}$.

Theorem 2.3. *The sequence $X_n(\cdot)$ converges weakly to $X(\cdot)$, in $\mathcal{D}([0,\infty) : I\!R_+)$, as $n \to \infty$.*

We will omit the proof of this theorem since it is very similar (in fact simpler) to the corresponding proof for the motor/cargo coupled system which is presented in the next section.

An important numerical quantity of interest for biological motors is their asymptotic velocity. The following theorem shows that, under appropriate periodicity conditions on the coefficients, the diffusion ratchet model for a biological motor given in Definition 2.2 has a well defined asymptotic velocity.

Theorem 2.4. *Assume that the coefficients a and b, in addition to the Lipschitz condition and* (2.4) *satisfy the periodicity conditions*

$$a(x + iL) = a(x), \;\; b(x + iL) = b(x), \;\; \forall\, x \in [0, L), \; i \in I\!N_0. \tag{2.10}$$

Then $\frac{X(t)}{t}$ converges almost surely to $\nu \doteq \frac{L}{I\!E(\tau^{(0)})}$, where $\tau^{(0)}$ is as defined in (2.5).

Remark 2.5. *The periodicity condition in the above theorem can be relaxed to the assumption that there exist functions $\tilde{b} : [0, L] \to I\!R$, $\tilde{a} : [0, L] \to I\!R$ such that as $i \to \infty$*

$$\sup_{0 \le x \le L} \Big(|b(x + iL) - \tilde{b}(x)| + |a(x + iL) - \tilde{a}(x)| \Big) \to 0.$$

In this case $\tau^{(0)}$ in the above theorem is given via (2.3) *and* (2.5) *with a, b there replaced by $\tilde{a}, \tilde{b}$.*

Proof of Theorem 2.4. The periodicity assumption gives that $\{\tau^{(i)}\}_{i \in I\!N_0}$ defined in (2.5) is an i.i.d. sequence. This yields that $v(t) \doteq \max\{m : \sum_{i=0}^{m-1} \tau^{(i)} \le t\}$ is a renewal process. The result now follows on observing that $|\frac{X(t)}{t} - \frac{v(t)L}{t}| \le L\frac{1}{t}$, $I\!E(\tau^{(0)}) < \infty$ (See Theorem 5.1) and the LLN for renewal processes (cf. Theorem 1.7.3 of [3]). ■

Theorem 2.4 shows that in order to compute the asymptotic velocity it suffices to compute the expected value of a certain first passage time. Typically this expected value does not have a simple closed form solution and one needs numerical methods for its computation. However, in the special case where $b(x) \equiv \mu$ and $a(x) \equiv \sigma$ one can give an exact closed form expression for the asymptotic velocity as the following result shows. The value of the asymptotic velocity obtained below agrees with results obtained in biophysics literature via formal limiting analysis of Fokker-Planck equations associated with "imperfect ratchets"[4].

Theorem 2.6. *Suppose that $b(x) = \mu$ and $a(x) = \sigma$ for all $x \in I\!R_+$. Then*

$$\lim_{t\to\infty} \frac{X(t)}{t} = \begin{cases} \frac{D}{L}\frac{\omega_l^2}{e^{\omega_l}-1-\omega_l} & \text{if } \mu \neq 0 \\ \frac{2D}{L}. & \text{if } \mu = 0, \end{cases} \tag{2.11}$$

where $D = \frac{\sigma^2}{2}$ and $\omega_l = -\frac{2\mu}{\sigma^2}L$.

Sketch of the Proof. The proof is a consequence of the fact that the Laplace transform of $\tau^{(0)}$: $\phi(\lambda) \doteq E_0[e^{-\lambda\tau^{(0)}}]$, is given as (cf. [7], Chapter 5):

$$\phi(\lambda) = \frac{\alpha+\beta}{\beta e^{-\alpha L} + \alpha e^{\beta L}} \tag{2.12}$$

where

$$\alpha \equiv \alpha(\lambda) = \frac{\sqrt{\mu^2+2\sigma^2\lambda}}{\sigma^2} + \frac{\mu}{\sigma^2}, \quad \beta \equiv \beta(\lambda) = \frac{\sqrt{\mu^2+2\sigma^2\lambda}}{\sigma^2} - \frac{\mu}{\sigma^2}.$$

The result now follows on using Theorem 2.4 and taking limit of $\phi'(\lambda)$ as $\lambda \to 0$. ∎

The following theorem gives a functional central limit theorem for fluctuations of $\frac{X(t)}{t}$ about the asymptotic velocity.

Theorem 2.7. *Under the assumptions of Theorem 2.4,*

$$\frac{1}{\sqrt{n}}\left(X(n\cdot) - \frac{L}{m}\cdot\right) \text{ converges weakly to } \frac{Ls}{m^{3/2}}B(\cdot)$$

in $C([0,\infty) : I\!R)$ as $n \to \infty$, where $m = I\!E\tau^{(0)}$, $s^2 = Var(\tau^{(0)})$, and B is a standard Brownian motion.

The above theorem yields the effective diffusivity of the biological motor as $\frac{L^2s^2}{m^3}$.

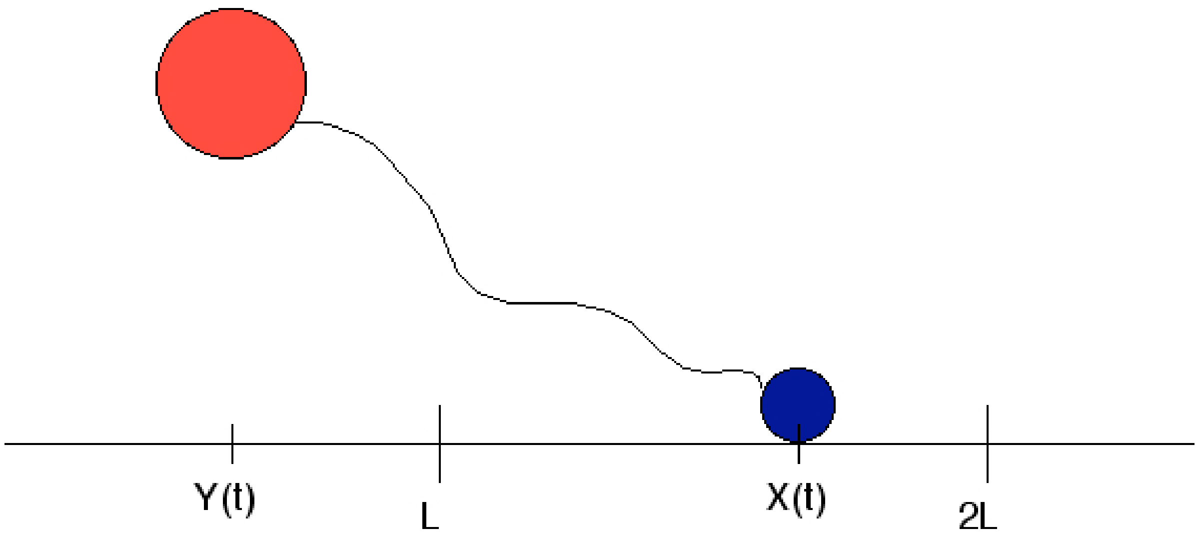
Proof. Note that $X(t) = Lv(t) + \epsilon_t$ where $v(\cdot)$ is as in the proof of Theorem 2.4 and $\epsilon_t = X(t) - \lfloor\frac{X(t)}{L}\rfloor L$. Thus, $X(nt) = Lv(nt) + \epsilon_{nt}$. Centering and normalizing, we obtain

$$\frac{m^{3/2}}{Ls\sqrt{n}}\left(X(nt) - \frac{Lnt}{m}\right) = \frac{v(nt) - \frac{nt}{m}}{sm^{-3/2}\sqrt{n}} + \frac{\epsilon_{nt}}{Lsm^{-3/2}\sqrt{n}}.$$

Since $0 \le \epsilon_{nt} < L$, $\sup_{0\le u\le t}\left|\frac{\epsilon_{nsu}}{Lsm^{-3/2}\sqrt{n}}\right| \to 0$ for all t with probability one as $n \to \infty$. The first term on the right side converges weakly to a standard Brownian motion by Theorem 14.6 of [2]. This proves the result. ∎

3 The Coupled System: Motor and Cargo.

In this section, we introduce a model for the dynamics of a biological motor which is pulling a cargo linked to the motor via a protein strand. As noted in the introduction, since the motor moves on a straight track along the x-axis, it suffices to describe the x- coordinate dynamics of the cargo. The evolution of the motor-cargo pair is given as follows. Once more, ratchet sites are located on the track at equally spaced intervals of length L; when the motor is at a "non-ratchet" site it can either move to the left or to the right in steps of size $\frac{1}{n}$. However, when the motor is at a ratchet site, it can only move to the right. Here n is, as before, a scaling parameter taking values in $I\!N'$.



The cargo, (or more precisely, "the projection of its location on the x -axis")in contrast to the motor, is free to move to the left or to the right, at every site in $\tilde{\mathbb{S}}_n \doteq \{\frac{j}{n} : j \in \mathbb{Z}\}$. More precisely, denoting the location of the motor at time t by $X_n(t)$ and that of the (x-coordinate of the) cargo by $Y_n(t)$, the pair (X_n, Y_n) is a Jump-Markov process with state space $\overline{\mathbb{S}}_n \doteq \mathbb{S}_n \times \tilde{\mathbb{S}}_n$ and infinitesimal generator $\mathfrak{q}_n \equiv \{\mathfrak{q}_n(z, \tilde{z}) : z, \tilde{z} \in \overline{\mathbb{S}}_n\}$ given as follows. Let $e_1 \doteq (1, 0)$ and $e_2 \doteq (0, 1)$. For $z = (x, y) \in \overline{\mathbb{S}}_n$:

$$\mathfrak{q}_n(z, z + n^{-1}e_1) = n^2\alpha_1(z) + n\beta_{11}(z); \quad \mathfrak{q}_n(z, z - n^{-1}e_1) = (n^2\alpha_1(z) + n\beta_{12}(z))1_{x\in\mathbb{S}_n\setminus\mathbb{L}}$$
$$\mathfrak{q}_n(z, z + n^{-1}e_2) = n^2\alpha_2(z) + n\beta_{21}(z); \qquad \mathfrak{q}_n(z, z - n^{-1}e_2) = n^2\alpha_2(z) + n\beta_{22}(z).$$

We set $\mathfrak{q}_n(z, z) \doteq -\sum_{i=1}^{2}(\mathfrak{q}_n(z, z + \frac{1}{n}e_i) + \mathfrak{q}_n(z, z - \frac{1}{n}e_i))$. For all remaining $z, \tilde{z} \in \overline{\mathbb{S}}_n$, $\mathfrak{q}_n(z, \tilde{z})$ is set to be 0. In the above α_i, β_{ij}, $i, j = 1, 2$ are Lipschitz functions from $I\!R^+ \times I\!R \to I\!R^+$. The dynamics of this Markov process can be described as follows. Given that the process is at $z = (x, y)$ at time t, the waiting time to the next transition is exponentially distributed with rate $-\mathfrak{q}_n(z, z) = (\lambda_n^1(z) + \lambda_n^2(z))$, where for $i = 1, 2$ $\lambda_n^i(z) \doteq \mathfrak{q}_n(z, z + \frac{1}{n}e_i) + \mathfrak{q}_n(z, z - \frac{1}{n}e_i)$. At the end of sojourn time, there is a either a transition in the y coordinate (with probability

$\frac{\lambda_n^2(z)}{\lambda_n^1(z)+\lambda_n^2(z)}$); otherwise, there is a transition in the x coordinate. If the transition is in the y coordinate, the y component increases by $\frac{1}{n}$ with probability $p_n^2(z) = \frac{\mathfrak{q}_n(z,z+\frac{1}{n}e_2)}{\lambda_n^2}$ and decreases with probability $1-p_n^2(z)$. Similarly, if the transition is in the x component, the x coordinate increases by $\frac{1}{n}$ with probability $p_n^1(z)$ and decreases by $\frac{1}{n}$ with probability $1-p_n^1(z)$, where $p_n^1(z) = \frac{\mathfrak{q}_n(z,z+\frac{1}{n}e_1)}{\lambda_n^1}$. Note that if x is a ratchet site, i.e. $x \in \mathbb{L}$, $p_n^1(z) = 1$. Thus in this case the probability of x coordinate moving to the left is 0.

Our next step will be to study the diffusion limit of the above Markov chain, as $n \to \infty$. In the limit, one would expect to obtain a diffusion ratchet, representing the dynamics of the biological motor, which is coupled with an unconstrained diffusion process representing the cargo. More precisely, we will prove that as $n \to \infty$, (X_n, Y_n) converges weakly, in $D([0,\infty) : I\!\!R_+ \times I\!\!R)$ to the process (X, Y), with paths in $C([0,\infty) : I\!\!R_+ \times I\!\!R)$, defined as follows.

Definition 3.1 (Motor and Cargo). *The Motor-Cargo pair is a stochastic process (X, Y) with sample paths in $C([0,\infty) : I\!\!R_+ \times I\!\!R)$, defined as follows*

$$\begin{cases} Y^{(i)}(t) &= Y^{(i)}(0) + \int_0^t b_2(X^{(i)}(s), Y^{(i)}(s))ds + \int_0^t a_2(X^{(i)}(s), Y^{(i)}(s))dB^{(i)}(s),\ t \geq 0 \\ X^{(i)}(t) &= \Gamma_i\left(iL + \int_0^\cdot b_1(X^{(i)}(s), Y^{(i)}(s))ds + \int_0^\cdot a_1(X^{(i)}(s), Y^{(i)}(s))dW^{(i)}(s)\right)(t),\ t \geq 0 \\ \tau^{(i)} &\doteq \inf\{t : X^{(i)}(t) = (i+1)L\},\ \ \sigma^{(i)} \doteq \tau^{(i-1)} + \sigma^{(i-1)},\ \ \sigma^{(0)} = 0 \\ X(t) &\doteq X^{(i)}(t-\sigma^{(i)}),\ \ Y(t) \doteq Y^{(i)}(t-\sigma^{(i)}),\ \ t \in [\sigma^{(i)}, \sigma^{(i+1)}),\ \ i \in I\!\!N_0 \\ Y^{(i)}(0) &\doteq Y^{(i-1)}(\tau^{(i)}) \text{ for } i \geq 1,\ \ Y^{(0)}(0) \doteq y_0. \end{cases} \tag{3.13}$$

where $W^{(i)}$ and $B^{(i)}$ are sequences of independent Wiener processes defined on some probability space $(\Omega, \mathcal{F}, I\!\!P)$, Γ_i is defined via (2.2). It is assumed that for all $(x,y) \in I\!\!R_+ \times I\!\!R$, $b_i(x,y)$ is uniformly bounded for $i = 1,2$ and $0 < a_* \leq a_i(x,y) \leq a^*$ for $i = 1,2$. Also, $b_i(\cdot,\cdot)$ and $a_i(\cdot,\cdot)$ are assumed to be globally Lipschitz continuous.

Note that in contrast to the "motor only" case $(X^{(i)}, Y^{(i)})$ is not independent of $(X^{(i+1)}, Y^{(i+1)})$ since they are related through the initial condition of $Y^{(i+1)}$. For some of the proofs and calculations, we need to define an "unreflected" form of $X^{(i)}$

$$U^{(i)}(t) \doteq iL + \int_0^t b_1(X^{(i)}(s), Y^{(i)}(s))ds + \int_0^t a_1(X^{(i)}(s), Y^{(i)}(s))dW^{(i)}(s). \tag{3.14}$$

To ensure that the above definition of the stochastic process (X,Y) is well-defined for all $t \in [0,\infty)$, we have the following lemma.

Lemma 3.2. *For all $i \in I\!\!N_0$, $\sigma^{(i)} \in (0,\infty)$ with probability one and $\sigma^{(i)} \to \infty$ almost surely as $i \to \infty$.*

Proof. Proof of the first part of the lemma is identical to that of Lemma 2.1. For the second part, note that for an arbitrary $\delta \in (0,1)$, we have as in (2.9) that for all j in $I\!\!N$

$I\!P[\tau^{(j)} \le \delta | \mathcal{F}_j] \le \frac{C\delta^{1/2}}{L}$, where $\mathcal{F}_j = \hat{\mathcal{F}}_{\sigma^{(j)}}$ and $\hat{\mathcal{F}}_t = \sigma\{X(s), Y(s) : s \le t\}$. As a consequence, we have for δ small enough,

$$\sum_{j=0}^{\infty} I\!P[\tau^{(j)} \ge \delta | \mathcal{F}_j] = \infty \text{ a.s.}$$

From the Borel-Cantelli Lemma (see Corollary 3.2 of Chapter 4 of [3]), we now have that $I\!P[\tau^{(j)} \ge \delta$ for infinitely many $j] = 1$. This proves the result. ■

The following theorem is the main result of this section. Let $b_i, a_i,\ i = 1, 2$ be as in Definition 3.1. Set, for $i = 1, 2$, $\beta_{i1} \doteq b_i^+$, $\beta_{i2} \doteq b_i^-$ and $\alpha_i \doteq \frac{1}{2}a_i^2$. Recall the Jump-Markov process, (X_n, Y_n), introduced at the beginning of the section. Let the initial condition of the process be δ_{0,y_n}, where $y_n \in \tilde{\mathbb{S}}_n$ and $y_n \to y_0$ as $n \to \infty$.

Theorem 3.3. *The sequence (X_n, Y_n) converges weakly to (X, Y), in $\mathcal{D}([0,\infty) : I\!R_+ \times I\!R)$, as $n \to \infty$.*

The key steps in the proof are the following two lemmas. Let

$$\mathcal{X}_0 \doteq \mathcal{D}([0,\infty) : I\!R_+) \times \mathcal{D}([0,\infty) : I\!R) \times [0,\infty],$$

where $[0,\infty]$ denotes the one point compactification of $I\!R_+$. Let $\mathcal{X} \doteq \mathcal{X}_0^{\otimes\infty}$. We will endow $\mathcal{X}$ with the usual topology and consider the Borel σ-field $\mathcal{B}(\mathcal{X})$. Define

$$\begin{aligned} \tilde{\mathcal{X}} \ &\doteq \{(x_i, y_i, \beta_i)_{i \in I\!N_0} \in \mathcal{X} | 0 < \beta_i < \infty, x_i \in \mathbb{A}, y_i \in C([0,\infty) : I\!R) \ \ \forall\, i\ , \\ &\qquad \text{and } \textstyle\sum_{i=0}^{j} \beta_i \to \infty \text{ as } j \to \infty\} \end{aligned} \tag{3.15}$$

where $\mathbb{A}$ is defined as

$$\begin{aligned} \mathbb{A} \ &\doteq \ \{\phi \in C([0,\infty) : I\!R_+) : \forall \delta > 0, \exists \text{ some } t' \in [\tau(\phi), \tau(\phi) + \delta] \\ &\qquad \text{such that } \phi(t') > L\}, \end{aligned} \tag{3.16}$$

with

$$\tau(\phi) \doteq \inf\{t : \phi(t) \ge L\}. \tag{3.17}$$

Lemma 3.4. *Let τ be defined via (3.17). Let $\phi \in \mathbb{A}$ and $\{\phi_n\}$ be a sequence in $D([0,\infty) : I\!R_+)$ such that $\phi_n \to \phi$. Then $\tau(\phi_n) \to \tau(\phi)$.*

For $\zeta = \{x^{(i)}, y^{(i)}, \tau^{(i)}\}_{i \in I\!N_0} \in \mathcal{X}$, define $(x_\zeta, y_\zeta) \in \mathcal{D}([0,\infty) : I\!R_+ \times I\!R)$ as:

$$x_\zeta(t) \doteq x^{(i)}(t - \sigma^{(i)}), \ \ y_\zeta(t) \doteq y^{(i)}(t - \sigma^{(i)}) \ \text{ if } \ t \in [\sigma^{(i)}, \sigma^{(i+1)}); \ \ i \in I\!N_0 \tag{3.18}$$

where $\sigma^{(i)} = \sum_{j=0}^{i-1} \tau^{(j)}$ for $i \ge 1$, $\sigma^{(0)} \doteq 0$.

Lemma 3.5. *Let $\Psi : \mathcal{X} \to \mathcal{D}([0,\infty) : I\!R_+ \times I\!R)$ be defined as $\Psi(\zeta) = (x_\zeta, y_\zeta)$, $\zeta = \{x^{(i)}, y^{(i)}, \tau^{(i)}\}_{i \ge 0}$, where (x_ζ, y_ζ) is given via (3.18). Then Ψ is continuous at every $\zeta \in \tilde{\mathcal{X}}$.*

Proof of Lemma 3.4 is contained in that of Theorem 9.4.3 of [9]. Lemma 3.5 will be proved after Theorem 3.3.

Proof of Theorem 3.3. To prove the theorem, we will use a somewhat more convenient representation (in distribution) for (X_n, Y_n), given as follows. Let $\lambda \doteq \sup_n \sup_{z\in\bar{\mathbb{S}}_n} n^{-2}|\mathfrak{q}_n(z,z)|$. Define a transition probability function $\mu'_n : (\bar{\mathbb{S}}_n, \mathcal{B}(\bar{\mathbb{S}}_n)) \to [0,1]$ as

$$\begin{aligned}\mu'_n(z,\Lambda) &= \left(1+\frac{\mathfrak{q}_n(z,z)}{n^2\lambda}\right)\delta_z(\Lambda) + \frac{\lambda^1_n(z)}{n^2\lambda}\left[p^1_n(z)\delta_{z+\frac{1}{n}e_1}(\Lambda) + (1-p^1_n(z))\delta_{z-\frac{1}{n}e_1}(\Lambda)\right]\\ &+ \frac{\lambda^2_n(z)}{n^2\lambda}\left[p^2_n(z)\delta_{z+\frac{1}{n}e_2}(\Lambda) + (1-p^2_n(z))\delta_{z-\frac{1}{n}e_2}(\Lambda)\right].\end{aligned}$$

Now, let $\mathcal{Y}^n \equiv (\mathcal{Y}^n_1, \mathcal{Y}^n_2)$ be a Markov chain with transition function μ'. Let $V(t)$ be a unit rate Poisson process which is independent of $\mathcal{Y}^n$. Then it can be checked (cf. Section 4.2 of [5])

$$(X_n(t), Y_n(t))_{t\geq 0} \stackrel{\mathcal{L}}{=} (\mathcal{Y}^n_1(V(\lambda n^2 t)), \mathcal{Y}^n_2(V(\lambda n^2 t)))_{t\geq 0}$$

From Proposition 4.5 of [8] it follows that (X_n, Y_n) converges weakly in $\mathcal{D}([0,\infty) : I\!R_+ \times I\!R)$ iff $(X^*_n(\cdot), Y^*_n(\cdot)) \doteq (\mathcal{Y}^n_1(\lfloor \lambda n^2 \cdot \rfloor), \mathcal{Y}^n_2(\lfloor \lambda n^2 \cdot \rfloor))$ converges weakly in $\mathcal{D}([0,\infty) : I\!R_+ \times I\!R)$ in which case the limits are the same. Thus to prove the result it suffices to show that (X^*_n, Y^*_n) converges weakly to (X, Y), in $\mathcal{D}([0,\infty) : I\!R_+ \times I\!R)$, as $n \to \infty$. In order to explicitly bring out the reflection mechanism in the prelimit process X^*_n, we define, for each $n \in I\!N'$, a family of processes $\{(\tilde{X}^{(i)}_n, \tilde{Y}^{(i)}_n, \tilde{U}^{(i)}_n)\}_{i\in I\!N_0}$ and stopping times $\{\tau^{(i)}_n\}_{i\in I\!N_0}$ recursively as follows. For $i = 0$,

$$(\tilde{X}^{(i)}_n(t), \tilde{Y}^{(i)}_n(t), \tilde{U}^{(i)}_n(t)) \doteq \Xi^{(i)}_n(\lfloor n^2\lambda t\rfloor)$$

where $\{\Xi^{(i)}_n(k)\}_{k\in I\!N_0}$ is a discrete space Markov chain with state space $\mathbb{G}_n \equiv \mathbb{S}_n \times \tilde{\mathbb{S}}_n \times \tilde{\mathbb{S}}_n$ and transition function $\hat{\mu}^{(i)}_n$ given as follows. Define $\psi_i : \mathbb{G}_n \to \mathbb{G}_n$ as $\psi_i((x,y,u)) \doteq (iL + (x-1/n-iL)^+, y, u-1/n)$. Set $\hat{e}_1 = (1,0,1)$ and $\hat{e}_2 = (0,1,0)$. Let λ^2_n and p^2_n be as given above Definition 3.1. Also define $\hat{\mathfrak{q}}_n(z, z-\frac{1}{n}e_1) \doteq n^2\alpha_1(z) + n\beta_{12}(z)$, $\hat{\lambda}^1_n(z) \doteq \mathfrak{q}_n(z, z+\frac{1}{n}e_1) + \hat{\mathfrak{q}}_n(z, z-\frac{1}{n}e_1)$, $\hat{p}^1_n(z) = \frac{\mathfrak{q}_n(z,z+\frac{1}{n}e_1)}{\hat{\lambda}^1_n}$ and $-\hat{\mathfrak{q}}_n(z,,z) = (\hat{\lambda}^1_n(z) + \lambda^2_n(z))$. For $\xi \equiv (x,y,u) \equiv (z,u) \in \mathbb{G}_n$ and $\Lambda \subset \mathbb{G}_n$

$$\begin{aligned}\hat{\mu}^{(i)}_n(\xi,\Lambda) &\doteq \left(1+\frac{\hat{\mathfrak{q}}_n(z,z)}{n^2\lambda}\right)\delta_\xi(\Lambda) + \frac{\hat{\lambda}^1_n(z)}{n^2\lambda}\left[\hat{p}^1_n(z)\delta_{\xi+\frac{1}{n}\hat{e}_1}(\Lambda) + (1-\hat{p}^1_n(z))\delta_{\psi_i(\xi)}(\Lambda)\right]\\ &+ \frac{\lambda^2_n(z)}{n^2\lambda}\left[p^2_n(z)\delta_{\xi+\frac{1}{n}\hat{e}_2}(\Lambda) + (1-p^2_n(x,y))\delta_{\xi-\frac{1}{n}\hat{e}_2}(\Lambda)\right], \qquad (3.19)\end{aligned}$$

where the initial condition of the above Markov chain is $\delta_{(0,y_n,0)}$. Also, define

$$\tau^{(i)}_n = \inf\{t : \tilde{X}^{(i)}_n(t) = (i+1)L\} \qquad (3.20)$$

Having defined $(\tilde{X}^{(j)}_n(t), \tilde{Y}^{(j)}_n(t), \tilde{U}^{(j)}_n(t))$ for $j = 1,\ldots,i-1$, define for $j = i$,

$$(\tilde{X}^{(i)}_n(t), \tilde{Y}^{(i)}_n(t), \tilde{U}^{(i)}_n(t)) \doteq \Xi^{(i)}_n(\lfloor n^2\lambda t\rfloor)$$

where $\{\Xi_n^{(i)}(k)\}_{k\in I\!N_0}$ is as before the discrete space Markov chain with state space $\mathbb{G}_n$ and with transition function $\hat{\mu}_n^{(i)}$ defined via (3.19). The initial condition of this Markov chain is $\delta_{(iL,\tilde{Y}_n^{(i-1)}(\tau_n^{(i-1)}),iL)}$ where $\tau_n^{(i-1)}$ is defined via (3.20). Set $\sigma_n^{(i)} = \sum_{j=0}^{i-1} \tau_n^{(j)}$.

Now, define

$$(\hat{X}_n(t), \hat{Y}_n(t)) = (\tilde{X}_n^{(i)}(t-\sigma_n^{(i)}), \tilde{Y}_n^{(i)}(t-\sigma_n^{(i)})); \;\; t \in [\sigma_n^{(i)}, \sigma_n^{(i+1)}); \;\; i \in I\!N_0.$$

By construction, $\tilde{X}_n^{(i)} = \Gamma_i(\tilde{U}_n^{(i)})$, $i \in I\!N_0$. Furthermore, $(\hat{X}_n, \hat{Y}_n)$ has the same law as (X_n^*, Y_n^*). So, it suffices to show that $(\hat{X}_n, \hat{Y}_n) \Rightarrow (X, Y)$, as $n \to \infty$. The advantage of working with $(\hat{X}_n, \hat{Y}_n)$ rather than (X_n^*, Y_n^*) is that the dynamical description of the former is very similar to that of (X, Y) given in Definition 3.1. In particular $(\hat{X}_n, \hat{Y}_n) = \Psi(Z_n)$ and $(X, Y) = \Psi(Z)$ where $\mathcal{Z}_n \equiv \{\mathcal{Z}_n(i)\}_{i\in I\!N_0}$ and $\mathcal{Z} \equiv \{\mathcal{Z}(i)\}_{i\in I\!N_0}$ are $\mathcal{X}$ valued random variables, where $\mathcal{X}$ is defined above (3.15) and

$$\mathcal{Z}_n(i) \doteq (\tilde{X}_n^{(i)}, \tilde{Y}_n^{(i)}, \tau_n^{(i)}) \quad \text{and} \quad \mathcal{Z}(i) \doteq (X^{(i)}, Y^{(i)}, \tau^{(i)}) \tag{3.21}$$

Furthermore, since the diffusion coefficient $a_1(x, y)$ is uniformly non-degenerate, $X^{(i)} \in \mathbb{A}$ for each i. We see from this fact and Lemma 3.2 that

$$I\!P(\mathcal{Z} \in \tilde{\mathcal{X}}) = 1. \tag{3.22}$$

The key step in the proof of Theorem 3.3 is to establish the weak convergence of $\mathcal{Z}_n$ to $\mathcal{Z}$ as $n \to \infty$. Observing that, conditioned on $\tilde{Y}_n^{(i-1)}(\tau_n^{(i-1)})$, $\mathcal{Z}_n(i)$ is independent of $\{\mathcal{Z}_n(j), j < i\}$ and $(\tilde{X}_n^{(i-1)}(\tau_n^{(i-1)}), \tilde{Y}_n^{(i-1)}(\tau_n^{(i-1)})) = (\tilde{X}_n^{(i)}(0), \tilde{Y}_n^{(i)}(0))$, we have that for convergence of $\mathcal{Z}_n \to \mathcal{Z}$ it suffices to show that $\mathcal{Z}_n(i) \to \mathcal{Z}(i)$ for each i. We will proceed inductively.

Induction Step. Assume that for some $i \in I\!N$, $\mathcal{Z}_n(i-1) \Rightarrow \mathcal{Z}(i-1)$. We will now show that $\mathcal{Z}_n(i) \Rightarrow \mathcal{Z}(i)$.

Using the Skorokhod representation theorem we can assume without loss of generality that $\mathcal{Z}_n(i-1) \to \mathcal{Z}(i-1)$ almost surely. Note that since $\tilde{Y}^{(i-1)}$ has continuous paths almost surely we have that $\tilde{Y}_n^{(i-1)} \to \tilde{Y}^{(i-1)}$ uniformly on compact intervals. Also, since $\tau_n^{(i-1)} \to \tau^{(i-1)}$ and $\tau^{(i-1)} < \infty$ a.s., it follows that $\tilde{Y}_n^{(i-1)}(\tau_n^{(i-1)}) \to \tilde{Y}^{(i-1)}(\tau^{(i-1)})$. Noting that $\tilde{Y}_n^{(i)}(0) = \tilde{Y}_n^{(i-1)}(\tau_n^{(i-1)})$, we have that $\{\tilde{Y}_n^{(i)}(0)\}_{n\in I\!N'}$ is a tight family. Since $\tilde{X}_n^{(i)}(0) = \tilde{U}_n^{(i)}(0) = iL$, $\{\tilde{X}_n^{(i)}(0), \tilde{U}_n^{(i)}(0)\}_{n\in I\!N'}$ is also a tight family. In what follows once more, we will suppress i from the notation when needed. Define $N(t) \doteq \lfloor n^2\lambda t \rfloor$ and $\Delta_j \tilde{Y}_n \doteq \tilde{Y}_n(\frac{j+1}{n^2\lambda}) - \tilde{Y}_n(\frac{j}{n^2\lambda})$. Define $\Delta_j \tilde{U}_n$ similarly. Define for $t \geq 0$,

$$\tilde{w}_n^{(i)}(t) \doteq \sum_{\ell=0}^{N(t)-1} \frac{[\Delta_\ell \tilde{U}_n^{(i)} - I\!E_\ell^n \Delta_\ell \tilde{U}_n^{(i)}]}{a_1(\tilde{X}_n^{(i)}(\frac{\ell}{n^2\lambda}), \tilde{Y}_n^{(i)}(\frac{\ell}{n^2\lambda}))}, \;\; \tilde{B}_n^{(i)}(t) \doteq \sum_{\ell=0}^{N(t)-1} \frac{[\Delta_\ell \tilde{Y}_n^{(i)} - I\!E_\ell^n \Delta_\ell \tilde{Y}_n^{(i)}]}{a_2(\tilde{X}_n^{(i)}(\frac{\ell}{n^2\lambda}), \tilde{Y}_n^{(i)}(\frac{\ell}{n^2\lambda}))}.$$

Here $I\!E_j^n$ is the expected value conditioned on $\mathcal{F}(\tilde{X}_n(s), \tilde{Y}_n(s), \tilde{U}_n(s), s \leq \frac{j}{n^2\lambda})$. We will next show that $\{(\tilde{X}_n, \tilde{U}_n, \tilde{Y}_n, \tilde{w}_n, \tilde{B}_n)\}_{n\in I\!N'}$ is tight. The tightness of $\tilde{U}_n$, $\tilde{Y}_n$, $\tilde{w}_n$, and $\tilde{B}_n$ will follow as a consequence of the Aldous-Kurtz criterion for tightness in the Skorohod space (see Theorem 2.1 in Section 9.2 of [9]), and the tightness of $\tilde{X}_n$ will be an immediate consequence of the continuity of the Skorokhod map, Γ_i.

Using (3.19), it is easy to check that the following "local consistency" conditions hold. Let $x' = \tilde{X}_n(\frac{j}{n^2\lambda}), y' = \tilde{Y}_n(\frac{j}{n^2\lambda})$.

$$\begin{aligned}
&I\!\!E_j^n \Delta_j \tilde{U}_n = b_1(x',y')(n^2\lambda)^{-1} + O(n^{-3}), \quad I\!\!E_j^n \Delta_j \tilde{Y}_n = b_2(x',y')(n^2\lambda)^{-1} + O(n^{-3}) \\
&\mathrm{Var}_j^n(\Delta_j \tilde{U}_n) = a_1^2(x',y')(n^2\lambda)^{-1} + O(n^{-3}), \quad \mathrm{Var}_j^n(\Delta_j \tilde{Y}_n) = a_2^2(x',y')(n^2\lambda)^{-1} + O(n^{-3}), \\
&\mathrm{Covar}_j^n(\Delta_j \tilde{U}_n, \Delta_j \tilde{Y}_n) = O(n^{-3}).
\end{aligned} \tag{3.23}$$

where $O(n^{-3})$ is a quantity which is bounded in absolute value by $\frac{C}{n^3}$ where C is a universal constant.

Using these local consistency conditions, we can establish the following inequality in a straightforward manner

$$I\!\!E^n|\tilde{U}_n(t) - \tilde{U}_n(0)|^2 \le 2\left|Kt + N(t)O(n^{-3})\right|^2 + 2\left(K^2 t + N(t)O(n^{-3})\right), \tag{3.24}$$

where K is the bound for the maximum of $|b_1^+|$, $|b_1^-|$ and $|a_1|$. Observing that $N(t) \le n^2\lambda t$, we have that the first condition of the Aldous-Kurtz criterion (Theorem 2.1 in Section 9.2 of [9]) is satisfied.

For the second condition of the Aldous-Kurtz criterion, fix $T > 0$ and take an arbitrary stopping time ς s.t. $\varsigma \le T$ $w.p.1$. Note that for $\delta' > 0$

$$\begin{aligned}
I\!\!E^n(1 \wedge |\tilde{U}_n(\varsigma+\delta') - \tilde{U}_n(\varsigma)|) &\le (I\!\!E^n|\tilde{U}_n(\varsigma+\delta') - \tilde{U}_n(\varsigma)|^2)^{1/2} \\
&\le \Bigg(2\left|K\delta' + N(\delta')O(n^{-3})\right|^2 \\
&\quad +2\left(K^2\delta' + N(\delta')O(n^{-3})\right)\Bigg)^{\frac{1}{2}},
\end{aligned}$$

where K is a universal constant. Since the right side above converges to 0 as $\delta' \to 0$ and $n \to \infty$, we have that the second condition of the Aldous-Kurtz criterion holds. This shows that $\{\tilde{U}_n\}$ is tight. The tightness of $\tilde{X}_n$ is now an immediate consequence of the fact that $\tilde{X}_n = \Gamma_i(\tilde{U}_n)$ and that Γ_i is a continuous map. Proof of tightness of $\{\tilde{Y}_n, w_n, B_n\}$ is very similar. In particular, note that (3.24) follows with $\tilde{U}_n$ replaced by $\tilde{Y}_n$ in exactly the same manner. This along with the already proved tightness of $\{\tilde{Y}_n(0)\}$ gives that the first condition in the Aldous-Kurtz criterion is satisfied. The remainder of the proof is virtually identical to that for $\tilde{U}_n$.

Denote the measure induced (on an appropriate path space) by $(\tilde{X}_n, \tilde{U}_n, \tilde{Y}_n, \tilde{w}_n, \tilde{B}_n)$ by $\mathbb{Q}_n$. The above tightness shows that every subsequence of $\mathbb{Q}_n$ admits a convergent sequence. So, in order to prove that $\mathcal{Z}_n(i) \Rightarrow \mathcal{Z}(i)$, it suffices to show that for any weakly convergent subsequence, $\{n'\}$, the weak limit of $(\tilde{X}^{(i)}_{n'}, \tilde{U}^{(i)}_{n'}, \tilde{Y}^{(i)}_{n'}, \tilde{w}^{(i)}_{n'}, \tilde{B}^{(i)}_{n'})$, denoted by $(\tilde{X}^{(i)}, \tilde{U}^{(i)}, \tilde{Y}^{(i)}, \tilde{W}^{(i)}, \tilde{B}^{(i)})$ has the same law as $(X^{(i)}, U^{(i)}, Y^{(i)}, W^{(i)}, B^{(i)})$ defined in Definition 3.1. We will use n for n' to simplify notation and as before, evoking the Skorokhod representation theorem, assume without loss of generality that the convergence is almost sure. Using standard martingale characterization results (cf. Theorem 9.4.2 of [9]) and the local consistency conditions in (3.23) one can easily show that $(\tilde{W}^{(i)}, \tilde{B}^{(i)})$ are independent Brownian motions with respect to the filtration $\mathcal{F}^{(i)}_t \doteq \sigma\{\tilde{X}^{(i)}(s), \tilde{U}^{(i)}(s), \tilde{Y}^{(i)}(s), \tilde{W}^{(i)}(s), \tilde{B}^{(i)}(s),\ s \le t\}$.

Next, we identify the limit process $(\tilde{X}, \tilde{U}, \tilde{Y})$. For each $\delta > 0$ and $t \in [j\delta, (j+1)\delta)$, define $\tilde{U}^{\delta}_n(t) \doteq \tilde{U}_n(j\delta)$, $\tilde{U}^{\delta}(t) \doteq \tilde{U}(j\delta)$. Processes $\tilde{X}^{\delta}_n, \tilde{X}^{\delta}, \tilde{Y}^{\delta}_n, \tilde{Y}^{\delta}$ are defined in a similar way. We set $\tilde{Z}^{\delta}_n \doteq (\tilde{X}^{\delta}_n, \tilde{Y}^{\delta}_n)$; $\tilde{Z}^{\delta}$ is defined similarly. Also, let $N^{\delta}(t) = \lfloor t/\delta \rfloor$. Using the definition of $\tilde{w}_n(\cdot)$ and the local consistency properties (3.23), it follows that for each $t \in [0, \infty)$

$$\tilde{U}^{\delta}_n(t) - \tilde{U}_n(0) = \int_0^t b_1(\tilde{Z}^{\delta}_n(s))ds + \sum_{j=0}^{N(t)-1} a_1(\tilde{Z}^{\delta}_n(\delta j))[\tilde{w}_n(\delta(j+1)) - \tilde{w}_n(\delta j)] + \epsilon^{\delta,t}_n,$$

where $I\!\!E \sup_{0\le s\le t} |\epsilon^{\delta,s}_n| \to 0$ as $\delta \to 0$, uniformly in n. A similar representation holds for $\tilde{Y}^{\delta}_n(t) - \tilde{Y}_n(0)$ with b_1, a_1 above replaced by b_2, a_2 and $\tilde{w}_n$ replaced by $\tilde{B}_n$. Since $(\tilde{X}_n, \tilde{U}_n, \tilde{Y}_n) \to (\tilde{X}, \tilde{U}, \tilde{Y})$, we have that $(\tilde{X}^{\delta}_n, \tilde{U}^{\delta}_n, \tilde{Y}^{\delta}_n) \to (\tilde{X}^{\delta}, \tilde{U}^{\delta}, \tilde{Y}^{\delta})$ in the D-space. Letting $n \to \infty$ in the above display, we have

$$\tilde{U}^{\delta}(t) - \tilde{U}(0) = \int_0^t b_1(\tilde{Z}^{\delta}(s))ds + \sum_{j=0}^{N(t)-1} a_1(\tilde{Z}^{\delta}(\delta j))[\tilde{W}(\delta(j+1)) - \tilde{W}(\delta j)] + O(\delta) + \epsilon^{\delta,t},$$

where $I\!\!E|\epsilon^{\delta,t}| \to 0$ as $\delta \to 0$. So, using the boundedness of the coefficients we have

$$\tilde{U}^{\delta}(t) - \tilde{U}(0) = \int_0^t b_1(\tilde{Z}^{\delta}(s))ds + \int_0^t a_1(\tilde{Z}^{\delta}(s))d\tilde{W}(s) + \overline{\epsilon_{\delta,t}},$$

where $I\!\!E|\overline{\epsilon_{\delta,t}}| \to 0$ as $\delta \to 0$. By the continuity of $a_i(\cdot), i = 1, 2$, $\tilde{X}(\cdot)$, and $\tilde{Y}(\cdot)$, we have that $\int_0^t a_1(\tilde{X}^{\delta}(s), \tilde{Y}^{\delta}(s))d\tilde{W}(s) \to \int_0^t a_1(\tilde{X}(s), \tilde{Y}(s))d\tilde{W}(s)$ in probability, as $\delta \to 0$. Similarly, $\int_0^t b_i(\tilde{X}^{\delta}(s), \tilde{Y}^{\delta}(s))ds \to \int_0^t b_i(\tilde{X}(s), \tilde{Y}(s))ds$ for $i = 1, 2$. Furthermore, since $\tilde{X}_n = \Gamma(\tilde{U}_n)$ for all n, we have that $\tilde{X} = \Gamma(\tilde{U})$. Therefore, the limit process $(\tilde{X}, \tilde{U})$ solves

$$\tilde{U}(t) = \tilde{U}(0) + \int_0^t b_1(\tilde{X}(s), \tilde{Y}(s))ds + \int_0^t a_1(\tilde{X}(s), \tilde{Y}(s))d\tilde{W}(s), \quad \tilde{X}(t) = \Gamma(\tilde{U})(t).$$

Exactly the same way one shows that $\tilde{Y}$ solves

$$\tilde{Y}(t) = \tilde{Y}(0) + \int_0^t b_2(\tilde{X}(s), \tilde{Y}(s))ds + \int_0^t a_2(\tilde{X}(s), \tilde{Y}(s))d\tilde{B}(s).$$

Noting that $\tilde{Y}_n^{(i)}(0) = \tilde{Y}_n^{(i-1)}(\tau_n^{(i-1)})$ and recalling that $\tilde{Y}_n^{(i-1)}(\tau_n^{(i-1)}) \to \tilde{Y}^{(i-1)}(\tau^{(i-1)})$, we have $\tilde{Y}^{(i)}(0) = \tilde{Y}^{(i-1)}(\tau^{(i-1)})$.

By strong (and weak) uniqueness of the solution to the above equations, we now have that $(\tilde{X}, \tilde{U}, \tilde{W}, \tilde{Y}, \tilde{B})$ has the same probability law as $(X^{(i)}, U^{(i)}, W^{(i)}, Y^{(i)}, B^{(i)})$. Finally note that $X^{(i)} \in \mathbb{A}$ a.s., where $\mathbb{A}$ is defined in 3.16. Thus, we have on using Lemma 3.4 that $\mathcal{Z}_n(i) \to \mathcal{Z}(i)$. This completes the proof of the "Induction Step". Since $\tilde{Y}_n^{(0)}(0) = y_n \to y_0$ as $n \to \infty$, one can show exactly as in the proof of the induction step that $\mathcal{Z}_n(0) \Rightarrow \mathcal{Z}(0)$. Thus $\mathcal{Z}_n(i) \to \mathcal{Z}(i)$ for each i. As noted earlier in the proof, an immediate consequence of the above results is that $\mathcal{Z}_n \to \mathcal{Z}$ as $n \to \infty$. The theorem now follows as an immediate consequence of Lemma 3.5, (3.22) and the continuous mapping theorem on observing that $(\hat{X}_n, \hat{Y}_n) = \Psi(\mathcal{Z}_n)$, $(X, Y) = \Psi(\mathcal{Z})$, and recalling that $(\hat{X}_n, \hat{Y}_n) \stackrel{\mathcal{L}}{=} (X_n^*, Y_n^*) \stackrel{\mathcal{L}}{=} (X_n, Y_n)$. ■

We now give the proof of Lemma 3.5.

Proof of Lemma 3.5. Let $\zeta_n, \zeta \in \mathcal{X}$ and $s_n, s \in [0, \infty)$ be such that $\zeta_n \to \zeta$, $s_n \to s$ and $\zeta \in \tilde{\mathcal{X}}$. It suffices to show that $\Psi(\zeta_n)(s_n) \to \Psi(\zeta)(s)$ as $n \to \infty$. Let $i \in I\!N_0$ be such that $s \in [\sigma^{(i)}, \sigma^{(i+1)})$. For notational convenience, we will denote $(x_n^{(j)}, y_n^{(j)})$ and $(x^{(j)}, y^{(j)})$ by z_n^j and z^j respectively. Consider first the case when $s \in (\sigma^{(i)}, \sigma^{(i+1)})$. In this case, we can assume, without loss of generality, that $s_n \in (\sigma_n^{(i)}, \sigma_n^{(i+1)})$ for all n. Thus

$$\Psi(\zeta_n)(s_n) = z_n^i(s_n - \sigma_n^{(i)}) \to z^i(s - \sigma^{(i)}) = \Psi(\zeta)(s),$$

where the convergence in the display follows from the convergence of $z_n^i \to z^i$ in the $\mathcal{D}$ space, continuity of z^i and convergence of $(s_n, \sigma_n^{(i)})$ to $(s, \sigma^{(i)})$. Finally consider the case where $s = \sigma^{(i)}$. Note that in this case

$$\Psi(\zeta)(s) = z^i(0) = z^{i-1}(\tau^{(i-1)}). \tag{3.25}$$

Without loss of generality, we can assume that $s_n \in [\sigma^{(i)}, \sigma^{(i+1)})$ or $s_n \in [\sigma^{(i-1)}, \sigma^{(i)})$. In the former case $\Psi(\zeta_n)(s_n) = z_n^i(s_n - \sigma_n^{(i)})$ while in the latter case $\Psi(\zeta_n)(s_n) = z_n^{(i-1)}(s_n - \sigma_n^{(i-1)})$. Once more using the convergence of $z_n^j \to z^j$ for $j = i$ and $i-1$, convergence of $(s_n, \sigma_n^{(i)}, \sigma_n^{(i-1)})$ to $(s, \sigma^{(i)}, \sigma^{(i-1)})$ and recalling (3.25), we see that $\Psi(\zeta_n)(s_n) \to \Psi(\zeta)(s)$. ■

4 Asymptotic Velocity of a Motor Pulling a Cargo.

In this section, we study the problem of existence of asymptotic velocity of a motor-cargo pair whose dynamics is modeled via a system of SDEs given as in Definition 3.1. The coupled problem, unlike the "motor only" case studied in Section 2, does not have a natural regenerative structure that can be exploited to guarantee the existence of the limit $\frac{X(t)}{t}$ as $t \to \infty$. We will consider the special case where the linkage between the motor and cargo is given by a linear spring. More precisely, letting X(t) and Y(t) represent the location of the motor and the cargo, respectively; at time t the dynamics of $(X(t), Y(t))$ is given as in Definition 3.1 with $b_1(x,y)$ replaced by $-\beta_1(x-y)$ and $b_2(x,y)$ replaced by $-\beta_2(y-x)$, where $\beta_i \in (0,\infty), i = 1,2$. Furthermore, for the sake of simplicity we take $a_1(x,y) \equiv a_1$ and $a_2(x,y) \equiv a_2$ where $a_1, a_2 \in (0,\infty)$. Note that the drift coefficients do not satisfy the boundedness assumption made in Definition 3.1. However, by a slight modification of the proof, one can show that Lemma 3.2 holds for this choice of coefficients as well. One can also carry out the proof of Theorem 3.3 for this unbounded case by a suitable localization argument. In view of the space limitation, this result will be reported elsewhere. Using the strong Markov property of the Brownian motion, the dynamics of (X, Y) can be expressed somewhat more explicitly as follows. Fix $x \in [0, L), z \in I\!R$. These represent the initial position of the motor and the initial distance between the motor and cargo, respectively. Set $Z \doteq X - Y$.

$$
\begin{aligned}
dX(t) &= -\beta_1 Z(t)dt + a_1 dW_1(t) + dl(t),\ X(0 = x \\
dY(t) &= +\beta_2 Z(t)dt + a_2 dW_2(t),\ Y(0) = z + x \\
l(t) &\doteq \sum_{j=1}^{J(t)-1} l_j + L_{J(t)-1}(t);\ l_j \doteq L_{j-1}(\sigma_j) \\
J(t) &\doteq \inf\{i : \sigma_i \geq t\},\ \sigma_i = \inf\{t : X(t) \geq (i+1)L\} \text{ for } i \in I\!N, \sigma_0 \doteq 0 \\
L_j(t) &= -\inf_{\sigma_j \leq s \leq t} \left((-\beta_1 \int_{\sigma_j}^{s} Z(u)du + a_1(W_1(s) - W_1(\sigma_j))) \bigwedge 0 \right) I_{[\sigma_j,\infty)}(t).
\end{aligned}
$$

In the above representation W_1, W_2 are two independent Brownian motions. Notice that if $X(t)$ is greater than $Y(t)$ then there is a positive drift in the cargo dynamics and a negative drift in the motor dynamics. The reverse is true when $Y(t)$ is greater than $X(t)$. Thus one expects that the distribution of $Z(t)$ converges to a stationary distribution as $t \to \infty$. The evolution of $Z(t)$ is described by the following equation:

$$dZ(t) = -\beta Z(t)dt + \sigma dW(t) + dl(t); Z(0) = z, \tag{4.26}$$

where $\beta \doteq \beta_1 + \beta_2$, $\sigma \doteq \sqrt{a_1^2 + a_2^2}$, and W is a Brownian motion given as $W \doteq \frac{1}{\sqrt{a_1^2+a_2^2}}(a_1^2 W_1 + a_2^2 W_2)$.

The main result of this section shows that the asymptotic velocity of the motor is well-defined; namely, the quantity $\frac{X(t)}{t}$ converges in probability to a deterministic value. Note that

$$\frac{X(t)}{t} = \frac{y}{t} - \frac{\beta_2}{t}\int_0^t Z(s)ds + a_2\frac{W_2(t)}{t} + \frac{Z(t)}{t}, \tag{4.27}$$

where $y \doteq z + x$. We will show that $\frac{Z(t)}{t} \to 0$ in probability and $\frac{1}{t}\int_0^t Z(s)ds$ converges in probability to a deterministic constant μ as $t \to \infty$. This will yield the desired result, namely, $\text{l.i.p.}_{t\to\infty}\frac{X(t)}{t} = -\beta_2\mu$, where l.i.p. denotes limit in probability. The key lemma in the proof of the result is the following. For a continuous function $\phi : [0,\infty) \to I\!R$, let $|\phi|_{*,t} \doteq \sup_{0\leq s\leq t}|\phi(s)|$.

Lemma 4.1. *There exist $c, \alpha, b, \kappa, \delta_0 \in (0,\infty)$ such that for all $z \in I\!R, x \in [0,L)$ and all $\Delta \le \delta_0$,*

$$I\!E_{x,z}|Z(s)-z|^2_{*,\Delta} \le C\Delta(1+\Delta z^2),\ I\!E_{x,z}Z^2(\Delta) - z^2 \le -\alpha\Delta z^2 + bI_{\{|z|<\kappa\}}, \tag{4.28}$$

where $I\!E_{x,z}$ refers to expectation with respect to probability measure under which $X(0) = x$ and $Y(0) = x+z$ a.s.

Remark It is easy to check that $(L\lfloor\frac{X(t)}{L}\rfloor, Z(t))$ is a Markov process with respect to $\mathcal{F}_t \doteq \sigma\{X(s), Y(s) : s \le t\}$. Ergodic properties of this Markov process will play a critical role in the analysis below.

The proof of Lemma 4.1 will be given at the end of this section. We begin by observing the following important consequence of the Lemma.

Lemma 4.2. *For all $\Delta \le \delta_0$ and $\pi \equiv (x,z) \in [0,L) \times I\!R$, there exist d_1, d_2 (possibly depending on Δ) such that*

$$\limsup_{n\to\infty} I\!E_\pi(Z^2(n\Delta)) < d_1|\pi|^2 + d_2.$$

Proof. Let $\Pi(t) \doteq (L\lfloor\frac{X(t)}{L}\rfloor, Z(t))$. Then observing that the first component of $\Pi(t)$ takes values in a compact set $[0,L]$, we have from Lemma 4.1 that for some $\tilde{\alpha}, \tilde{b}, \tilde{\kappa} \in (0,\infty)$,

$$I\!E_\pi|\Pi(n\Delta)|^2 - |\pi|^2 \le -\tilde{\alpha}|\pi|^2 + \tilde{b}I_{\{|\pi|\le\tilde{\kappa}\}}, \tag{4.29}$$

for all $\pi \in [0,L] \times I\!R$. Noting that (4.29) implies the condition (V2) (from page 262 of [10]), we have from Theorem 12.3.4 of [10] that the Markov chain $\{\Pi(n\Delta)\}_{n\ge1}$ has at least one invariant measure. Due to the non-degeneracy of the diffusion coefficients a_1, a_2, this Markov chain, $\{\Pi(n\Delta)\}_{n\ge1}$, is ψ-irreducible in the sense of Section 4.2 of [10] for $\psi = \lambda_1 \otimes \lambda_2$ where λ_1 is the normalized Lebesgue measure on $[0,L]$ and λ_2 is the standard normal measure on $I\!R$. Furthermore the chain is strongly aperiodic in the sense of Section 5.4.3 of [10]. This shows that the chain, $\{\Pi(n\Delta)\}_{n\ge1}$, has a unique invariant measure. Denote this measure by ν_Δ.

From (4.29), we have that condition (V3) from page 337 of [10] is satisfied with $f(\pi) = \frac{\alpha}{2}\Delta|z|^2 + 1$. From Theorem 14.2.6 and Proposition 14.3.1 [10], we then obtain

$$I\!E_\pi(Z(n\Delta)^2) \to \int_{[0,L]\times I\!R} z^2 \nu_\Delta(dx,dz) \text{ as } n \to \infty. \tag{4.30}$$

Furthermore from Theorem 14.2.3(i) of [10] and the representation for the invariant measure in Theorem 10.0.1 of [10], we obtain

$$\int_{[0,L]\times I\!R} z^2 \nu_\Delta(dx,dz) \le \frac{2|\pi|^2}{\alpha\Delta} + C_3, \tag{4.31}$$

where $C_3 \in (0,\infty)$ is independent of Δ. Combining (4.30) and (4.31), we obtain the result. ■

Lemma 4.3. *For all* $\pi \equiv (x,z) \in [0,L] \times I\!R$, $\frac{Z(t)}{t} \to 0$ *in* $L^1(I\!P_\pi)$ *as* $t \to \infty$.

Proof. Note that

$$\frac{Z(t)}{t} = \frac{Z(\Delta\lfloor \frac{t}{\Delta} \rfloor)}{t} + \frac{Z(t) - Z(\Delta\lfloor \frac{t}{\Delta} \rfloor)}{t}$$

Thus,

$$\limsup_{t\to\infty} I\!E_\pi \left| \frac{Z(t)}{t} \right| \le \limsup_{t\to\infty} \frac{I\!E_\pi |Z(\Delta\lfloor \frac{t}{\Delta} \rfloor)|}{t} + \limsup_{t\to\infty} \frac{I\!E_\pi |Z(t) - Z(\Delta\lfloor \frac{t}{\Delta} \rfloor)|}{t}.$$

From Lemma 4.2, we have that the first term above equals zero. Also, from the first inequality in Lemma 4.1, we have that

$$\limsup_{t\to\infty} \frac{I\!E_\pi |Z(t) - Z(\Delta\lfloor \frac{t}{\Delta} \rfloor)|}{t} \le \limsup_{t\to\infty} \frac{\sqrt{C\Delta(1 + \Delta I\!E|Z(\Delta\lfloor \frac{t}{\Delta} \rfloor)|^2}}{t}.$$

Applying Lemma 4.2 again, we see that the expression on the right side equals 0. This proves the result. ■

Proposition 4.4. *The Markov process* $\{\Pi(t)\}_{t\ge 0}$ *admits a unique invariant measure:* ν.

Proof. It suffices to show that the family of probability measures, $\{\tilde{\nu}_t, t \ge 0\}$, where

$$\tilde{\nu}_t(A) \doteq \frac{1}{t} \int_0^t I\!P_\pi[Z(s) \in A] ds, A \in \mathcal{B}(I\!R), \tag{4.32}$$

is tight. We will show that $\limsup_{t\to\infty} \int_{[0,L]\times I\!R} |\tilde{z}|^2 d\tilde{\nu}_t(d\tilde{x}, d\tilde{z}) < \infty$. This clearly will prove the required tightness and hence prove the result. Note that

$$\int_{[0,L]\times I\!R} |\tilde{z}|^2 d\tilde{\nu}_t(d\tilde{x}, d\tilde{z}) = \frac{1}{t} \int_0^t I\!E_\pi |Z(s)|^2 ds. \tag{4.33}$$

Also, note that

$$I\!E_\pi |Z(s)|^2 \le 2I\!E_\pi |Z(\Delta\lfloor \frac{s}{\Delta} \rfloor)|^2 + 2I\!E_\pi |Z(s) - Z(\Delta\lfloor \frac{s}{\Delta} \rfloor)|^2 \tag{4.34}$$

$$\le 2I\!E_\pi |Z(\Delta\lfloor \frac{s}{\Delta} \rfloor)|^2 + 2C\Delta(1 + \Delta I\!E_\pi |Z(\Delta\lfloor \frac{t}{\Delta} \rfloor)|^2 \tag{4.35}$$

where the second inequality follows from Lemma 4.1. Substituting the above inequality into (4.33) and using Lemmas 4.1 and 4.2, we get

$$\limsup_{t\to\infty} \int_{[0,L]\times I\!R} |\tilde{z}|^2 d\nu_t(d\tilde{x}, d\tilde{z}) \le C\Delta + (1 + C\Delta^2)(d_1|z|^2 + d_2) \ < \infty. \tag{4.36}$$

This proves the result. ■

We now come to the main result of the chapter.

Theorem 4.5. *For all* $\pi \equiv (x,z) \in [0,L] \times I\!R$,

$$\frac{X(t)}{t} \to -\beta_2 \int_{[0,L]\times I\!R} z\nu(dx,dz)$$

in $L^1(I\!P_\pi)$ *as* $t \to \infty$*; where* ν *is as in Proposition 4.4.*

Proof. Recalling (4.27), we have from Lemma 4.3 that

$$\limsup_{t\to\infty} I\!E_\pi \left| \frac{X(t)}{t} + \beta_2 \int_{[0,L]\times I\!R} z\nu(du,dz) \right| = \beta_2 \limsup_{t\to\infty} I\!E_\pi \left| \frac{1}{t}\int_0^t Z(s)ds - \int_{[0,L]\times I\!R} z\nu(du,dz) \right|$$

From the ergodicity proven in Proposition 4.4, we have for all $k \in I\!N$

$$I\!E_\pi \left| \frac{1}{t}\int_0^t f_k(Z(s))ds - \int_{[0,L]\times I\!R} f_k(z)\nu(du,dz) \right| \to 0$$

as $t \to \infty$, where $f_k(z) = (z \vee k) \wedge (-k)$. The result will follow from the dominated convergence theorem if we have that $\sup_t \frac{1}{t}\int_0^t I\!E_\pi |Z(s)|^2 ds < \infty$. However, this an immediate consequence of (4.36) which was established in the proof of Proposition 4.4. This proves the result. ■

We will now provide the proof of Lemma 4.1 which was critically used in the proof of the above theorem. In what follows $c_1, c_2, \ldots$ will denote generic constants whose values change from one proof to the next. We begin by obtaining some preliminary bounds.

In order to bound $I\!EZ^2(\Delta) - z^2$, we first obtain a bound on the "reflection term" as follows:

$$l(t) \le \beta_1 t|Z|_{*,t} + a_1 \sum_{j=1}^{\infty} \underbrace{\sup_{\sigma_{j-1}\wedge t \le s \le \sigma_j \wedge t} |W_1(s) - W_1(\sigma_{j-1})|}_{M_j(t)}. \tag{4.37}$$

Writing

$$l(t) = \underbrace{l(t) - \beta_1 t|Z|_{*,t}}_{\tilde{l}(t)} + \beta_1 \underbrace{t|Z|_{*,t}}_{l^*(t)}, \tag{4.38}$$

we have from (4.37) that $\tilde{l}(t) \le a_1 \sum_{j=1}^{\infty} M_j(t)$. This bound enables us to prove the following inequality.

Lemma 4.6.

$$I\!E\tilde{l}^2(t) \le ct(\sqrt{I\!E[X^2(t)]} + 1). \tag{4.39}$$

Proof. Observe that

$$I\!E\tilde{l}^2(t) \le a_1^2 \sum_{j=1}^{\infty} I\!EM_j^2(t) + 2a_1^2 \sum_{k=1}^{\infty}\sum_{i=k+1}^{\infty} I\!EM_i(t)M_k(t). \tag{4.40}$$

From Burkholder-Grundy-Davis inequalities, we have that

$$I\!\!E M_j^2(t) \le c_1 I\!\!E(\sigma_j \wedge t - \sigma_{j-1} \wedge t), \tag{4.41}$$

where c_1 is a universal constant.

This immediately yields that

$$\sum_{j=1}^{\infty} I\!\!E M_j^2(t) \le c_1 I\!\!E \sum_{j=1}^{\infty} (\sigma_j \wedge t - \sigma_{j-1} \wedge t) = c_1 t. \tag{4.42}$$

Let $\mathcal{G}_t = \sigma\{W_i(s) : s \le t, i = 1, 2\}$, and set $\mathcal{F}_k \doteq \mathcal{G}_{\sigma_k}$. Then, recalling the definition of $J(t)$ in (4.26) we have

$$\begin{aligned} I\!\!E[\sum_{i=k+1}^{\infty} M_i(t)|\mathcal{F}_k] &= I\!\!E[\sum_{i=k+1}^{J(t)} M_i(t)|\mathcal{F}_k] \\ &\le I\!\!E[\left(\sum_{i=k+1}^{\infty} M_i^2(t)\right)^{1/2} J(t)^{1/2}|\mathcal{F}_k] \\ &\le \left(I\!\!E[\sum_{i=k+1}^{\infty} M_i^2(t)|\mathcal{F}_k]\right)^{1/2} (I\!\!E[J(t)|\mathcal{F}_k])^{1/2} \\ &\le \sqrt{c_1}\sqrt{t}\sqrt{I\!\!E[J(t)|\mathcal{F}_k]}, \end{aligned} \tag{4.43}$$

where in the last step we have once more used the Burkholder-Grundy-Davis inequalities. An immediate consequence of the above inequality is the following:

$$\begin{aligned} \sum_{k=1}^{\infty}\sum_{i=k+1}^{\infty} I\!\!E[M_k(t)M_i(t)] &= I\!\!E\sum_{k=1}^{\infty} M_k(t) I\!\!E[\sum_{i=k+1}^{\infty} M_i(t)|\mathcal{F}_k] \\ &\le \sqrt{c_1}\sqrt{t} I\!\!E \sum_{k=1}^{J(t)} M_k(t)\sqrt{I\!\!E[J(t)|\mathcal{F}_k]} \\ &\le \sqrt{c_1}\sqrt{t} I\!\!E\left(\left(\sum_{k=1}^{\infty} M_k^2(t)\right)^{1/2}\left(\sum_{k=1}^{\infty} 1_{\{k\le J(t)\}} I\!\!E[J(t)|\mathcal{F}_k]\right)^{1/2}\right) \\ &\le \sqrt{c_1}\sqrt{t}\left(I\!\!E\sum_{k=1}^{\infty} M_k^2(t)\right)^{1/2}\left(I\!\!E\sum_{k=1}^{\infty} 1_{\{k\le J(t)\}} I\!\!E[J(t)|\mathcal{F}_k]\right)^{1/2} \\ &\le c_1 t\left(I\!\!E\sum_{k=1}^{\infty} 1_{\{k\le J(t)\}} I\!\!E[J(t)|\mathcal{F}_k]\right)^{1/2}, \end{aligned} \tag{4.44}$$

where we have used (4.43) in the first inequality and (4.42) in the fourth inequality. On observing that the event $\{k \le J(t)\}$ is $\mathcal{F}_k$-measurable. We rewrite the right side of (4.44) as

$$c_1 t \left(I\!E \sum_{k=1}^{\infty} 1_{\{k \le J(t)\}} I\!E[J(t)|\mathcal{F}_k] \right)^{1/2} = c_1 t \left(I\!E \sum_{k=1}^{\infty} J(t) 1_{\{k \le J(t)\}} \right)^{1/2} = c_1 t \sqrt{I\!E J^2(t)}.$$

Substituting the above, (4.42) and (4.44) into (4.40), we have $I\!E\tilde{l}^2(t) \le c_2(t + 2t\sqrt{I\!E[J^2(t)]})$. Finally, observing that $J(t) \le \frac{X(t)+L}{L}$, we obtain from the above inequality that $I\!E\tilde{l}^2(t) \le ct(\sqrt{I\!E[X^2(t)]} + 1)$, for a suitable constant c. ■

Next we obtain a bound on $I\!E(X^2(t))$.

Lemma 4.7. *There exists a $\gamma_1 \in (0, \infty)$ such that $I\!E(X^2(t)) \le \gamma_1(1 + t^2 + t^2 I\!E|Z|^2_{*,t})$ for all $t \ge 0$.*

Proof. Recalling the representation of X in (4.26) and that $x \in [0, L]$, we have from Lemma 4.6 and (4.38) that

$$\begin{aligned} I\!E X^2(t) &\le c_1[L^2 + \beta_1^2 t^2 I\!E|Z|^2_{*,t} + t + t(\sqrt{I\!E[X^2(t)]} + 1)] \\ &\le c_2[1 + t + t^2 I\!E|Z|^2_{*,t} + t\sqrt{I\!E X^2(t)}]. \end{aligned} \tag{4.45}$$

Set $d_1 = c_2[1 + t + t^2 I\!E|Z|^2_{*,t}]$, $d_2 = c_2 t$, $I\!E X^2(t) \doteq \alpha^2$. Then (4.45) can be rewritten as $\alpha^2 \le d_1 + d_2\alpha$. This immediately yields that $\alpha^2 \le c_3(1 + t + t^2 I\!E|Z|^2_{*,t})$. This proves the result. ■

Combining Lemmas 4.6 and 4.7, we have that

$$I\!E\tilde{l}^2(t) \le t\sqrt{\gamma_1}(\sqrt{1 + t^2 + t^2 I\!E|Z|^2_{*,t}} + 1) \ \le \gamma_2(t + t^2 + t^2\sqrt{I\!E|Z|^2_{*,t}}), \tag{4.46}$$

for a suitable $\gamma_2 \in (0, \infty)$. Now, we proceed to the proof of Lemma 4.1.

Proof of Lemma 4.1. Define $\tilde{Z}(s) \doteq Z(s) - z$. Applying Itô's formula we have

$$\tilde{Z}^2(t) = -2\beta \int_0^t \tilde{Z}(s)Z(s)ds + 2\int_0^t \tilde{Z}(s)dW(s) + 2\int_0^t \tilde{Z}(s)dl(s) + \sigma^2 t.$$

Thus,

$$(1 - 2\beta\Delta)|\tilde{Z}|^2_{*,\Delta} \le 2|\int_0^{\cdot} \tilde{Z}(u)dW(u)|_{*,\Delta} + 2|\int_0^{\cdot} \tilde{Z}(u)dl(u)|_{*,\Delta} + \sigma^2\Delta + 2\Delta\beta|z||\tilde{Z}|_{*,\Delta}.$$

Taking expectations in the above inequality, we have

$$(1 - 2\beta\Delta)I\!E|\tilde{Z}|^2_{*,\Delta} \le 4\sqrt{\Delta}\sqrt{I\!E|\tilde{Z}|^2_{*,\Delta}} + 2I\!E(|\tilde{Z}|_{*,\Delta} l(\Delta)) + \sigma^2\Delta + 2\Delta\beta|z|I\!E|\tilde{Z}|_{*,\Delta}. \tag{4.47}$$

Furthermore, from (4.38) we have

$$\mathbb{E}(|\tilde{Z}|_{*,\Delta}l(\Delta)) \leq \mathbb{E}(|\tilde{Z}|_{*,\Delta}\tilde{l}(\Delta)) + \beta_1\Delta\mathbb{E}|\tilde{Z}|^2_{*,\Delta} + \beta_1\Delta|z|\mathbb{E}|\tilde{Z}|_{*,\Delta}. \tag{4.48}$$

Next, using (4.46) we get

$$\begin{aligned}
\mathbb{E}(|\tilde{Z}|_{*,\Delta}|\tilde{l}(\Delta)) &\leq (\mathbb{E}|\tilde{Z}|^2_{*,\Delta})^{\frac{1}{2}}(\mathbb{E}\tilde{l}^2(\Delta))^{\frac{1}{2}} \\
&\leq c_3(\mathbb{E}|\tilde{Z}|^2_{*,\Delta})^{\frac{1}{2}}\left(\Delta + \Delta^2 + \Delta^2\sqrt{\mathbb{E}|Z|^2_{*,\Delta}}\right)^{\frac{1}{2}} \\
&\leq c_4(\mathbb{E}|\tilde{Z}|^2_{*,\Delta})^{\frac{1}{2}}\left(\Delta + \Delta^2 + \Delta^2\sqrt{\mathbb{E}|\tilde{Z}|^2_{*,\Delta}} + |z|\Delta^2\right)^{\frac{1}{2}} \\
&\leq c_5(\mathbb{E}|\tilde{Z}|^2_{*,\Delta})^{\frac{1}{2}}\left(\Delta^{\frac{1}{2}} + \Delta + \Delta(\mathbb{E}|\tilde{Z}|^2_{*,\Delta})^{\frac{1}{4}} + \sqrt{|z|}\Delta\right).
\end{aligned} \tag{4.49}$$

Letting $\Theta = \mathbb{E}|\tilde{Z}|^2_{*,\Delta}$, we have from (4.47), (4.48), and (4.49) that

$$\begin{aligned}
(1-(2\beta+2\beta_1)\Delta)\Theta &\leq 4\sqrt{\Delta}\sqrt{\Theta} + (2\beta+2\beta_1)\Delta|z|\sqrt{\Theta} \\
&\quad +2c_5\sqrt{\Theta}(\sqrt{\Delta} + \Delta + \Delta\Theta^{\frac{1}{4}} + \Delta\sqrt{|z|}) + \sigma^2\Delta.
\end{aligned}$$

Renaming constants and dividing by $\sqrt{\Theta}$,

$$\begin{aligned}
(1-(2\beta+2\beta_1)\Delta)\sqrt{\Theta} &\leq (2\beta+2\beta_1)\Delta|z| + c_6(\sqrt{\Delta} + \Delta + \Delta\Theta^{\frac{1}{4}} + \Delta\sqrt{|z|}) + \frac{\sigma^2\Delta}{\sqrt{\Theta}} \\
&\leq c_7(\sqrt{\Delta} + \Delta\Theta^{\frac{1}{2}}) + \Delta(1+|z|)) + \frac{\sigma^2\Delta}{\sqrt{\Theta}}
\end{aligned}$$

Subtracting and absorbing constants, we have for $\Delta < 1$,

$$(1-(2\beta+2\beta_1+c_7)\Delta)\sqrt{\Theta} \leq c_8\sqrt{\Delta} + c_8\Delta|z| + \frac{\sigma^2\Delta}{\sqrt{\Theta}}.$$

Now choosing δ_0 sufficiently small, we obtain that for all $\Delta < \delta_0$,

$$\sqrt{\Theta} \leq c_9\left(\sqrt{\Delta}(1+\sqrt{\Delta}|z|) + \frac{\Delta}{\sqrt{\Theta}}\right).$$

Thus, $\Theta \leq c_9\left(\sqrt{\Delta}(1+\sqrt{\Delta}|z|)\sqrt{\Theta} + \Delta\right)$. Using the quadratic formula, we obtain that, for all $\Delta \leq \delta_0$,

$$\Theta \leq C\Delta(1+\Delta z^2), \tag{4.50}$$

for a suitable constant C. This proves the first inequality in Lemma 4.1.

Next, from (4.26), we have via another application of Itô's formula

$$\mathbb{E}Z^2(t\wedge\tau_k) = z^2 - 2\beta\mathbb{E}\int_0^{t\wedge\tau_k} Z^2(s)ds + 2\mathbb{E}\int_0^{t\wedge\tau_k} Z(s)dl(s) + \sigma^2\mathbb{E}(t\wedge\tau_k), \tag{4.51}$$

where $\tau_k \doteq \inf\{t : |Z(s)| \geq k\}$. Note that

$$\left|\int_0^{t\wedge\tau_k} Z(s)dl(s)\right| \leq (|Z|_{*,t})\, l(t) \ = (|Z|_{*,t})[\tilde{l}(t) + \beta_1 l^*(t)], \tag{4.52}$$

where the last equality follows from (4.38). Recalling the definition of $l^*(s)$,

$$I\!E[|Z|_{*,t} l^*(t)] \leq t I\!E|Z|^2_{*,t}. \tag{4.53}$$

Also, using (4.46), we have

$$I\!E|Z|_{*,t}\tilde{l}(t) \leq \sqrt{I\!E|Z|^2_{*,t}}\sqrt{I\!E\tilde{l}^2(t)} \ \leq c_1\sqrt{I\!E|Z|^2_{*,t}}\left(\sqrt{t} + t + t\left(I\!E|Z|^2_{*,t}\right)^{\frac{1}{4}}\right). \tag{4.54}$$

Now fix $\Delta \leq \delta_0$, and let $\alpha_\Delta \doteq I\!E|Z|^2_{*,\Delta}$. From (4.52), (4.53), and (4.54), we have on taking $k \to \infty$ in (4.51), that,

$$\begin{aligned} I\!E Z^2(\Delta) - z^2 \quad &\leq \quad -2\beta\Delta I\!E \inf_{0\leq u\leq\Delta}|Z(u)|^2 + 2\beta_1\Delta\alpha_\Delta \\ &\quad + c_2\sqrt{\Delta}\sqrt{\alpha_\Delta}(1 + \sqrt{\Delta} + \sqrt{\Delta}\,(\alpha_\Delta)^{\frac{1}{4}}) + \sigma^2\Delta. \end{aligned} \tag{4.55}$$

Next, noting that $|Z(u)| = |\tilde{Z}(u) + z|$, we have $Z^2(u) = \tilde{Z}^2(u) + z^2 + 2z\tilde{Z}(u)$. Thus,

$$I\!E \inf_{0\leq s\leq\Delta}|Z(u)|^2 \geq z^2 - 2|z|I\!E|\tilde{Z}|_{*,\Delta} \ \geq z^2 - 2|z|\sqrt{\Theta}. \tag{4.56}$$

Combining (4.55), (4.50), and (4.56)

$$\begin{aligned} I\!E Z^2(\Delta) - z^2 \quad &\leq \quad -2\beta\Delta(z^2 - 2|z|\sqrt{\Theta}) + 2\beta_1\Delta(z^2 + \Theta + 2|z|\sqrt{\Theta}) \\ &\quad + c_2\sqrt{\Delta}\sqrt{2z^2 + 2\Theta}(1 + \sqrt{\Delta} + \sqrt{\Delta}\left(2z^2 + 2\Theta\right)^{\frac{1}{4}}) + \sigma^2\Delta. \end{aligned}$$

Recalling the definition of Θ and renaming constants, we have

$$I\!E Z^2(\Delta) - z^2 \quad \leq \quad -2\beta_2\Delta z^2 + c_{10}(z^2 + 1) + c_{10}\Delta(|z|^{3/2} + 1).$$

The second inequality of Lemma 4.1 now follows with a sufficiently large value of κ , on choosing Δ sufficiently small. ■

5 Appendix

Theorem 5.1. *Let W be a standard Brownian motion given on some filtered probability space and let $x \in I\!R^+$. Define the process X^x as follows. For $t \geq 0$,*

$$X^x(t) \doteq \Gamma_0\left(x + \int_0^{\cdot} b(X^x(s))ds + \int_0^{\cdot} a(X^x(s))dW(s)\right)(t), \tag{5.57}$$

where b and a are Lipschitz continuous functions and $\inf_{x\in\mathbb{R}}|a(x)| \geq m > 0$. *Let* $L \in (0,\infty)$ *and define*

$$\tau^x \doteq \inf\{t : X^x(t) \notin [0,L)\}. \tag{5.58}$$

Then there exists an $\epsilon > 0$ *such that* $\mathbb{E}e^{\tau^x u} < \infty$ *for all* $-\epsilon < u < \epsilon$.

In order to prove the theorem, we begin with the following lemma.

Lemma 5.2. *Let* τ^x *be given by* (5.58). *Then* $\sup_{x\in[0,L]} \mathbb{P}[\tau^x > 1] < 1$.

Proof. We will argue by contradiction. Suppose that $\sup_{x\in[0,L]} \mathbb{P}[\tau^x > 1] = 1$. Then, there is a sequence $\{x_n\} \in [0,L]$ such that $\mathbb{P}[\tau^{x_n} > 1] \to 1$. Since $[0,L]$ is in a compact set, there is a convergent subsequence $\{x'_n\}$ which converges to some $x \in [0,L]$. We know that $\mathbb{P}[\tau^{x'_n} > 1] \to 1$. Now, if $P[\tau^y > 1]$ is a continuous function in y, then $\mathbb{P}[\tau^{x'_n} > 1] \to \mathbb{P}[\tau^x > 1]$ which implies that $\mathbb{P}[\tau^x > 1] = 1$. This in particular says that $\mathbb{P}(X^x(\frac{1}{2}) \in [0,L]) = 1$ which is clearly impossible in view of the uniform non-degeneracy of the diffusion coefficient. Thus we have a contradiction, which proves the lemma. Therefore it suffices to show that $\mathbb{P}[\tau^y > 1]$ is a continuous function of y. Note that $\mathbb{P}[\tau^y > 1] = \mathbb{P}(Z(y) < L)$, where $Z(y) \doteq \sup_{0\leq s\leq 1} X^y(s)$. Note that $Z(y_n) \to Z(y)$ in probability as $y_n \to y$. Finally, observing that the distribution of $Z(y)$ is absolutely continuous with respect to the Lebesgue measure on $[0,\infty)$, we have that $\mathbb{P}(Z(y_n) < L) \to \mathbb{P}(Z(y) < L)$ as $y_n \to y$. This proves that $\mathbb{P}[\tau^y > 1]$ is a continuous function of y.

Proof of Theorem 5.1. Let $\alpha \doteq \sup_{x\in[0,L]} \mathbb{P}[\tau^x > 1]$. From Lemma 5.2 we have that $\alpha \in (0,1)$. Now, fix $x \in [0,L]$ and suppress it from the notation.

$$\mathbb{P}[\tau > n] = \mathbb{E}\Big(I_{[\tau>n]}I_{[\tau>n-1]}\Big) = \mathbb{E}\Big(\mathbb{E}[I_{[\tau>n]}|\mathcal{F}_{n-1}]I_{[\tau>n-1]}\Big) \leq \alpha\mathbb{P}[\tau > n-1],$$

where $\mathcal{F}_t \doteq \sigma\{W(s) : 0 \leq s \leq t\}$. Thus, $\mathbb{P}[\tau > n] \leq \alpha^n$. This proves that $\mathbb{E}e^{\tau u} < \infty$ for all u such that $|u| < |\log(\alpha)|$. ■

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