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A STOCHASTIC RICHARDSON'S ARMS RACE MODEL

John Fricks
fricks@email.unc.edu
Department of Statistics
CB 3260
University of North Carolina
Chapel Hill NC 27599-3260
Randall Swift
rjswift@csupomona.edu
Department of Mathematics
California State Polytechnic University
Pomona CA 91768

SYNOPTIC ABSTRACT

A stochastic version of the Richardson's arms race model is considered through the method of birth-death processes. The expected value of the model is obtained and shown to be analogous to the original deterministic arms race model.

Key Words and Phrases: Richardson's Arms model; Birth-Death Process.

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1. INTRODUCTION

The Richardson's Arms Race Model has been studied widely since its introduction in the 1930's by its namesake Lewis F. Richardson. The model uses a pair of differential equations to represent the complex interaction that occurs between two nations involved in an arms build up. Richardson believed that the central cause of war was the excess of such arms and that studying the behaviors which change the levels of armaments may lead to insights that prevent war (Richardson 1960).

Richardson's basic model began with only two nations. Let us imagine there are nations X and Y who are neighbors (Richardson 1939). Since changes in X's and Y's war readiness may be an indicator of impending military action, we are interested in the contributing factors that will make these countries increase or decrease the amount of armaments that they own. We will use the variables x and y to denote the amounts of arms that X and Y respectively have. In light of these variable name choices, we will denote the rate of change in X's armaments over time as $\frac{dx}{dt}$ and the rate of change in Y's armaments as $\frac{dy}{dt}$. The basic model is as follows:

$$\begin{aligned} \frac{dx}{dt} &= \alpha y - \gamma x + \zeta \\ \frac{dy}{dt} &= \beta x - \delta + \eta \\ for \quad & \alpha,\ \beta,\ \gamma,\ \delta > 0. \end{aligned} \tag{1}$$

Looking only at the first equation, let us examine the source of each term on the right side. We see that the rate of change for country X's armament stock is positively proportional to Y's armament stock., representing "mutual fear" between nations. The rate of change in X's stock is negatively related to the amount of arms that it owns due to the fact that other things are being sacrificed by the nation in favor of arms. As this sacrifice increases, the nation will be less willing to give more to the procurement of arms. The last term accounts for feelings of good or ill will which are independent of armament stock. If this constant is positive, then country X has a greater propensity to increase its armament stock based only on its dislike for country Y. Similarly if the constant is negative, then country X has a positive opinion of country Y and is less prone to increase its arms.

The model is a rather simple representation of the actual complexities involved in an arms race between two countries. Richardson explained that this formulation was a reduction of an earlier model that used many more variables as well as square and interaction terms; he eventually decided that the elements above were the most valuable in explaining the situation while keeping a focus on simplicity. It should also be noted that Richardson allowed for the variables x and y to be negative. What does it mean to own a negative armament? To allow for a robust interpretation of his results, Richardson interpreted negative values for these variables as cooperation between the nations in terms of economic trade. Richardson examined the results of the model with differing values for the various coefficients, especially to determine if the model stabilizes. Michael Olnick summarizes these results succinctly in *An Introduction to Mathematical Models in the Social and Life Sciences* (1978). In addition to examaning the mathematical properties of the model, Richardson offered evidence to support his theories by using expenditure figures of European powers and describing how these expenditures related to the outbreak of war. The attempt to gather real world data in the support of his theories culminated in his work *The Statistics of Deadly Quarrels* (1960).

Since the model under investigation is an oversimplification, it is a natural extension to enrich the model by grouping the assorted effects not included explicitly in the model as a random influence on the model and then study the results. Previous attempts have been made at probabilistic extensions of the model. One of these extensions considers the deterioration of arms as a stochastic factor in the model (Gopsalmy 1977). Another approach was to include a uniform random variable in the model to account for all of the random perturbations (Mayer-Kress 1992). However, we will come at the problem from the perspective of a birth-death process. In the next section we will develop a stochastic model We then show that the expected value of the stochastic model presented corresponds to the original deterministic Richardson's model.

2. A STOCHASTIC MODEL

To begin with, we make some assumptions.Our model will assume that expenditures of countries X and Y are discrete valued; thus we will denote the expenditures of X and Y by the variables m and n, respectively.We introduce two interdependent random variables,M(t) and N(t),which represent are the possible discrete levels of expenditure of X and Y.Also,we need to break apart the grievance term into two components, one nonnegative and one nonpositive.This separation will make our model capable of dealing with the analog of the original model when we have negative values for these grievance terms.Unlike the Richardson model however, we will consider only nonnegative values of m and n.Our model is expressed as the probability of being at given levels of expenditure,m and n,at time t and is given by

$$\begin{aligned} P_{m,n}(t+\Delta t) &= (\alpha n+\rho)(\Delta t)P_{m-1,n}(t) \\ &\quad + (\gamma(m+1)+\kappa)(\Delta t)P_{m+1,n}(t) \\ &\quad +(\beta m+\tau)(\Delta t)P_{m,n-1}(t) \\ &\quad +(\delta(n+1)+\lambda)(\Delta t)P_{m,n+1}(t) \\ &\quad +(1-(\alpha n+\rho+\gamma m+\kappa+\beta m+\tau+\delta n+\lambda)(\Delta t))P_{m,n}(t), \\ &\quad \text{for } \alpha,\ \rho,\ \gamma,\ \kappa,\ \beta,\ \tau,\ \delta,\ \lambda > 0, \\ &\quad \text{where } P_{m,n}(t) = P\{M(t)=m, N(t)=n\}. \end{aligned} \tag{2}$$

As mentioned above, we have broken apart the grievance terms. So, the expression ρ-κ is analogous to the parameter ζ in the original model, and the expression $\tau-\lambda$ is analogous to the parameter η in the original model.

We will assume that in any given time step Δt only one event will happen. The probability that the model is in a particular state in a given time t+ Δt is the sum of probabilities of all of the cases which may occur during Δt. Since we are only allowing for one event in a given time step, there are five possible cases:

1. Country X increases its armaments
2. Country Y increases its armaments
3. Country X decreases its armaments
4. Country Y decreases its armaments
5. Nothing happens

Each of these terms consists of two parts. The first factor is the probability that a given event happens. The second factor is the probability that the system was in a state where this step could happen. We also assume that these events are independent, so that the probability of both occurring is the product of these factors.

We will now see how each of the terms in our model correlates to the original model.

1. $(\alpha n + \rho)(\Delta t)P_{m-1,n}(t)$– This term corresponds to the case that the model is at one less of m and at the same level of n in the previous time step. The first factor corresponds to the mutual fear term αy plus the positive contribution ρ from the grievance term ζ in the first differential equation of the original Richardson model (1). The second factor states that the contribution from this term is proportional to the length of the current time step Δt. The last factor is the probability that the model had arrived at this state at the previous time step.

2. $(\gamma(m+1)+\kappa)(\Delta t)P_{m+1,n}(t)$ – This term corresponds to the case that the model is at one more of m and at the same level of n in the previous time step, t. The first factor corresponds to the sum of a term γ(m+1) which corresponds to the drag term γx and the negative contribution κ which corresponds to the grievance term ζ in the first differential equation of the original Richardson model (1). Again, the second factor states that the contribution from this term is proportional to the length of the current time step Δt. The last factor is the probability that the model had arrived at this state at the previous time step t.

3. $(\beta m + \tau)(\Delta t)P_{m,n-1}(t)$– This term corresponds to the case that the model is at one less of n and at the same level of m in the previous time step t. The first factor corresponds to the mutual fear term βx plus the positive contribution τ of the grievance term η of the second differential equation of the original Richardson model (1). The second factor consists of the length of the time step Δt. The last factor is the probability that the model had arrived at this state at the previous time step t.

4. $(\delta(n+1)+\lambda)(\Delta t)P_{m,n+1}(t)$ –This term corresponds to the case that the model is at one more of n and at the same level of m in the previous time step t. The first factor is analogous to the drag term δy plus the negative contribution λ from the grievance term of the second differential equation in the original Richardson's model (1).

5.$(1 - (\alpha n + \rho + \gamma m + \kappa + \beta m + \tau + \delta n + \lambda)(\Delta t))P_{m,n}(t)$– This term considers the probability that nothing happens in this time step between t and t+Δt. Thus, we have one minus the sum of the probabilities of the other cases times the probability that we were originally in this state.

Simple equation manipulation gives

$$
\begin{aligned}
P_{m,n}(t+\Delta t) - P_{m,n}(t) = \; & (\alpha n + \rho)(\Delta t) P_{m-1,n}(t) \\
& + (\gamma(m+1) + \kappa)(\Delta t) P_{m+1,n}(t) \\
& + (\beta m + \tau)(\Delta t) P_{m,n-1}(t) \\
& + (\delta(n+1) + \lambda)(\Delta t) P_{m,n+1}(t) \\
-(\alpha n + \; \rho + \; & \gamma m + \kappa + \beta m + \tau + \delta n + \lambda)(\Delta t) P_{m,n}(t)
\end{aligned}
\tag{3}
$$

Dividing both sides by Δt, and letting Δt approach 0, gives

$$
\frac{dP_{m,n}(t)}{dt} = lim_{\Delta t \to 0} \frac{P_{m,n}(t+\Delta t) - P_{m,n}(t)}{dt} \tag{4}
$$

$$
\begin{aligned}
= \; & (\alpha n + \rho) P_{m-1,n}(t) \\
& + (\gamma(m+1) + \kappa) P_{m+1,n}(t) \\
& + (\beta m + \tau) P_{m,n-1}(t) \\
& + (\delta(n+1) + \lambda) P_{m,n+1}(t) \\
& - (\alpha n + \rho + \gamma m + \kappa + \beta m + \tau + \delta n + \lambda) P_{m,n}(t)
\end{aligned}
\tag{5}
$$

This gives an infinite system of differential equations (5). Often these types of systems may be solved recursively given an initial condition. This method seems to be untenable in this case, since there is recursion in two variables. However, a numerical solution for the system may be possible through the method of randomization, also know as uniformization. For the time being, we consider some deterministic properties of this stochastic model.

3. Deterministic Analysis of the Model

The probability generating function (pgf) is an aid in analyzing discrete valued random variables. The pgf ϕ of our model is given by

$$
\phi(t, z_1, z_2) = E[{z_1}^m {z_2}^n] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} {z_1}^m {z_2}^n P_{m,n}(t). \tag{6}
$$

Differentiation with respect to t yields

$$\partial_t \phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {z_1}^m {z_2}^n \frac{dP_{m,n}(t)}{d} t \tag{7}$$

Substitution from Equation (4) and separation of terms gives

$$\begin{aligned}
\partial_t \phi \quad = \quad & \alpha \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n {z_1}^m {z_2}^n P_{m-1,n}(t) \\
& + \rho \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {z_1}^m {z_2}^n P_{m-1,n}(t) \\
& + \gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (m+1) {z_1}^m {z_2}^n P_{m+1,n}(t) \\
& + \kappa \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {z_1}^m {z_2}^n P_{m+1,n}(t) \\
& + \beta \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m {z_1}^m {z_2}^n P_{m,n-1}(t) \\
& + \tau \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {z_1}^m {z_2}^n P_{m,n-1}(t) \\
& + \delta \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (n+1) {z_1}^m {z_2}^n P_{m,n+1}(t) \\
& + \lambda \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {z_1}^m {z_2}^n P_{m,n+1}(t) \\
& - \alpha \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n {z_1}^m {z_2}^n P_{m,n}(t) \\
& - \rho \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {z_1}^m {z_2}^n P_{m,n}(t) \\
& - \gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (m+1) {z_1}^m {z_2}^n P_{m,n}(t) \\
& - \kappa \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {z_1}^m {z_2}^n P_{m,n}(t) \\
& - \beta \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m {z_1}^m {z_2}^n P_{m,n}(t) \\
& - \tau \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {z_1}^m {z_2}^n P_{m,n}(t) \\
& - \delta \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (n+1) {z_1}^m {z_2}^n P_{m,n}(t)
\end{aligned} \tag{8}$$

$$-\lambda \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} {z_1}^m {z_2}^n P_{m,n}(t).$$

Rewriting some of the terms as a partial derivatives of ϕ gives

$$\begin{aligned} \partial_t \phi &= \alpha z_1 z_2 (\partial_{z_2} \phi) + \rho z_1 \phi + \gamma(\partial_{z_1} \phi) \\ &\quad + \frac{\kappa}{z_1}\phi + \beta z_1 z_2 (\partial_{z_1}\phi) + \tau z_2 \phi + \delta(\partial_{z_2}\phi) \\ &\quad + \frac{\lambda}{z_2}\phi - \alpha z_2(\partial_{z_2}\phi) - \rho\phi - \gamma z_1(\partial_{z_1}\phi) \\ &\quad -\kappa\phi - \beta z_1(\partial_{z_1}\phi) - \tau\phi - \delta z_2(\partial_{z_2}\phi) - \lambda\phi. \end{aligned} \tag{9}$$

We now group derivatives of ϕ:

$$\begin{aligned} \partial_t \phi &= (\beta z_1 z_2 + \gamma - \gamma z_1 - \beta z_1)(\partial_{z_1}\phi) \\ &\quad + (\alpha z_1 z_2 + \delta - \alpha z_2 - \delta z_2)(\partial_{z_2}\phi) \\ &\quad + (\rho z_1 + \frac{\kappa}{z_1} + \tau z_2 + \frac{\lambda}{z_2}\phi - \rho - \kappa - \tau - \lambda)\phi. \end{aligned} \tag{10}$$

To obtain the pgf ϕ of the System (4), a solution of Equation (9) is required. However, this partial differential equation is at best difficult to solve. It remains an open problem to find an explicit solution to Equation (9), though the expression in Equation (9) can be used to gather information about the system. Using the fact that the expected value of a random variable can be obtained from the derivative of its pgf, we take the derivative in Equation (9) with respect to z_1:

$$\begin{aligned} \partial_{z_1,t}\phi &= (\beta z_1 z_2 + \gamma - \gamma z_1 - \beta z_1)(\partial_{z_1,z_1}\phi) \\ &\quad (\beta z_2 - \gamma - \beta)(\partial_{z_1}\phi) \\ &\quad + (\alpha z_1 z_2 + \delta - \alpha z_2 - \delta z_2)(\partial_{z_1,z_2}\phi) \\ &\quad + (\alpha z_1)(\partial_{z_2}\phi) \\ &\quad + (\rho z_1 + \frac{\kappa}{z_1} + \tau z_2 + \frac{\lambda}{z_2} - \rho - \kappa - \tau - \lambda)(\partial_{z_1}\phi) \\ &\quad + (\rho - \kappa {z_1}^{-2})\phi. \end{aligned} \tag{11}$$

Now, $\partial_{z_1,t}\phi = \partial_{t,z_1}\phi$ since ϕ is continuous where it exists. So, letting $z_1 = 1$ and $z_2 = 1$ we obtain

$$\partial_{t,z_1}\phi(1,1) = \alpha(\partial_{z_2}\phi(1,1)) \tag{12}$$

$$-\gamma(\partial_{z_1}\phi(1,1))$$
$$+(\rho-\kappa)\phi(1,1).$$

From Equation (11) and the properties of a pgf we can write

$$\partial_t E[M(t)] = \alpha E[N(t)] - \gamma E[M(t)] + (\rho - \kappa). \tag{13}$$

Equation (12) corresponds with the first differential equation in the original model (1) the change in country X's expected arms $\partial_t E[M(t)]$ is equal to α times the expected amount of arms of country Y minus γ times the expected amount of arms of country X plus the goodwill term $\rho - \kappa$ which we defined to be ζ.

Similarly, we can take the derivative in Equation(9) with respect to z_2 to obtain the equivalent of the second equation in the original model:

$$\begin{aligned}
\partial_{z_2,t}\phi \;=\;& (\beta z_1 z_2 + \gamma - \gamma z_1 - \beta z_1)(\partial_{z_2,z_1}\phi) \\
& (\beta z_1)(\partial_{z_1}\phi) \\
& +(\alpha z_1 z_2 + \delta - \alpha z_2 - \delta z_2)(\partial_{z_2,z_2}\phi) \\
& +(\alpha z_1 - \alpha - \delta)(\partial_{z_2}\phi) \\
& +(\rho z_1 + \frac{\kappa}{z_1} + \tau z_2 + \frac{\lambda}{z_2} - \rho - \kappa - \tau - \lambda)(\partial_{z_2}\phi) \\
& +(\tau - \lambda {z_2}^{-2})\phi.
\end{aligned} \tag{14}$$

Again letting $z_1 = 1$ and $z_2 = 1$, we obtain

$$\begin{aligned}
\partial_{t,z_2}\phi(1,1) \;=\;& \beta(\partial_{z_1}\phi(1,1)) \\
& -\delta(\partial_{z_2}\phi(1,1)) \\
& +(\tau - \lambda)\phi(1,1).
\end{aligned} \tag{15}$$

Using a property of a pgf and the continuity of ϕ, we may rewrite Equation (14) a

$$\partial_t E[N(t)] = \beta E[M(t)] - \delta E[N(t)] + (\tau - \lambda). \tag{16}$$

As above, this equation agrees precisely with the second equation of the original model. The change in Y's expected armaments with respect to time $\partial_t E[N(t)]$ is equal to β times the expected amount of country X's armaments minus δ times the expected amount of country Y's armaments plus the goodwill term $\tau - \lambda$ which we defined to correspond to η.

It should now be clear that the model we have created in system (4) is indeed a stochastic version of the Richardson's arms race model. The expected value of our model corresponds to the deterministic original model.

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